

5D Diagonalizable Operators

5.48 definition: *diagonal matrix*

A diagonal matrix is a square matrix that is 0 everywhere except possibly on the diagonal.

$$\begin{bmatrix} * & * & 0 & 0 \\ 0 & a_{ii} & & \\ & & * & \\ 0 & & & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = a_{ii} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

5.50 definition: *diagonalizable*

An operator on V is called diagonalizable if the operator has a diagonal matrix with respect to some basis of V .

Notice:
If $T \in \mathcal{L}(V)$ is diagonalizable,
then $T v_k = M(T)_{kk} v_k$
for all v_k in the
basis $v_1, \dots, v_n$
diagonalizing $M(T)$

5.52 definition: *eigenspace, $E(\lambda, T)$*

Suppose $T \in \mathcal{L}(V)$ and $\lambda \in \mathbb{F}$. The eigenspace of T corresponding to λ is the subspace $E(\lambda, T)$ of V defined by

$$E(\lambda, T) = \text{null}(T - \lambda I) = \{v \in V : Tv = \lambda v\}.$$

Hence $E(\lambda, T)$ is the set of all eigenvectors of T corresponding to λ , along with the 0 vector. (i.e. $E(\lambda, T) \neq \{0\} \Leftrightarrow \lambda$ is an eigenvalue of T)

5.54 *sum of eigenspaces is a direct sum*

Suppose $T \in \mathcal{L}(V)$ and $\lambda_1, \dots, \lambda_m$ are distinct eigenvalues of T . Then

$$E(\lambda_1, T) + \dots + E(\lambda_m, T)$$

is a direct sum. Furthermore, if V is finite-dimensional, then

$$\dim E(\lambda_1, T) + \dots + \dim E(\lambda_m, T) \leq \dim V.$$

Good!

Proof To show $E(\lambda_1, T) + \dots + E(\lambda_m, T)$ is direct, suppose $v_1 + \dots + v_m = 0$ where $v_i \in E(\lambda_i, T)$, $i \in \{1, \dots, m\}$.

Since any nonzero v_i is an eigenvector of T corresponding to the eigenvalue λ_i , this is a sum of zero or more eigenvectors corresponding to distinct eigenvalues of T . But, eigenvectors corresponding to distinct eigenvalues are lin. indep., so this sum must have zero nonzero terms (i.e. $v_i = 0$ for $i \in \{1, \dots, m\}$). Hence, the sum is direct. Consequently, $\dim V \geq \dim(E(\lambda_1, T) + \dots + E(\lambda_m, T)) = \dim E(\lambda_1, T) + \dots + \dim E(\lambda_m, T)$.

5.55 conditions equivalent to diagonalizability

Suppose V is finite-dimensional and $T \in \mathcal{L}(V)$. Let $\lambda_1, \dots, \lambda_m$ denote the distinct eigenvalues of T . Then the following are equivalent.

- (a) T is diagonalizable.
- (b) V has a basis consisting of eigenvectors of T .
- (c) $V = E(\lambda_1, T) \oplus \dots \oplus E(\lambda_m, T)$.
- (d) $\dim V = \dim E(\lambda_1, T) + \dots + \dim E(\lambda_m, T)$.

Proof (a) $\Rightarrow$ (b) $\S$ T is diagonalizable. Let $v_1, \dots, v_n$ be a basis of V s.t. $M(T) = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$ for some $\lambda_1, \dots, \lambda_n \in \mathbb{F}$.
Then, by the definition of $M(T)$, $Tv_k = M(T)_{kk}v_k = \lambda_k v_k$. So, λ_k is an eigenvalue of T since $v_k \neq 0$ (and v_k is then an eigenvector).

(b) $\Rightarrow$ (c) $\S$ V has a basis consisting of e-vecs of T . Then, every $w \in V$ is a linear combo of e-vecs of T . Thus, it is a sum of elements from $E(\lambda_1, T), \dots, E(\lambda_m, T)$ where $\lambda_1, \dots, \lambda_m$ are all the distinct eigenvalues of T . So, $V = E(\lambda_1, T) + \dots + E(\lambda_m, T)$ and by our previous result this sum is direct.

(c) $\Rightarrow$ (d) $\S$ $V = E(\lambda_1, T) \oplus \dots \oplus E(\lambda_m, T)$. Then, we have $\dim V = \dim(E(\lambda_1, T) \oplus \dots \oplus E(\lambda_m, T)) = \dim E(\lambda_1, T) + \dots + \dim E(\lambda_m, T)$.

(d) $\Rightarrow$ (a) $\S$ $\dim V = \dim E(\lambda_1, T) + \dots + \dim E(\lambda_m, T)$. Take a basis for each $E(\lambda_i, T)$ & put them all together as a list $v_1, \dots, v_n$ w/ $n = \dim V$. $\S a_1, \dots, a_n \in \mathbb{F}$ s.t. $a_1 v_1 + \dots + a_n v_n = 0$. Write u_k for the portion of the sum belonging to $E(\lambda_k, T)$. So $u_1 + \dots + u_m = 0$. Since eigenvectors corresponding to distinct e-vals are lin. indep., and since each u_k is an eigenvector if $u_k \neq 0$, this must mean $u_k = 0$ for $k=1, \dots, m$. So, all the a_i must be zero as well. $\S$

5.58 enough eigenvalues implies diagonalizability

Suppose V is finite-dimensional and $T \in \mathcal{L}(V)$ has $\dim V$ distinct eigenvalues. Then T is diagonalizable.

$v_1, \dots, v_n$ is a lin. indep. list of length $\dim V$, & hence a basis of V .

Ex

Let $T \in \mathcal{L}(\mathbb{F}^3)$ defined by $T(x, y, z) = (2x + y, 5y + 3z, 8z)$.

Wrt. the basis $(1, 0, 0), (0, 1, 0), (0, 0, 1)$, $M(T) = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 5 & 3 \\ 0 & 0 & 8 \end{bmatrix}$.

This is not diagonal, but it shows that the eigenvalues of T are 2, 5, and 8. So, T has $\dim \mathbb{F}^3 = 3$ distinct eigenvalues. Hence, it can be diagonalized. To do this, just find an e-vector corresponding to each e-val here.

One choice: $\lambda_1 = 2 \rightsquigarrow v_1 = (1, 0, 0)$, $\lambda_2 = 5 \rightsquigarrow v_2 = (1, 3, 0)$, $\lambda_3 = 8 \rightsquigarrow v_3 = (1, 6, 1)$.

Then, if $v = a_1 v_1 + a_2 v_2 + a_3 v_3$, $Tv = a_1 T v_1 + a_2 T v_2 + a_3 T v_3$
 $= 2a_1 v_1 + 5a_2 v_2 + 8a_3 v_3$

So, this lets us know what T does to any vector $v \in V$ since v_1, v_2, v_3 is a basis. For instance, this makes computing powers of T easy!

$$T^k v = T^{k-1} (2a_1 v_1 + 5a_2 v_2 + 8a_3 v_3) = \dots = 2^k a_1 v_1 + 5^k a_2 v_2 + 8^k a_3 v_3$$