5.41 *determination of eigenvalues from upper-triangular matrix*

Suppose $T \in \mathcal{L}(V)$ has an upper-triangular matrix with respect to some basis of V. Then the eigenvalues of T are precisely the entries on the diagonal of that upper-triangular matrix.

$\boxed{\text{proof}}$ Suppose $v_1, \dots, v_n$ is a basis of V s.t. $T \in \mathcal{L}(V)$ has an UT matrix w.r.t. this basis.
Let $\lambda_1, \dots, \lambda_n$ be the diagonal entries of $\mathcal{M}(T)$, i.e.

$$\mathcal{M}(T) = \begin{bmatrix} \lambda_1 & & * \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$$

whatever

all 0's below

So, $Tv_1 = \lambda_1 v_1$. Hence, λ_1 is an eigenvalue of T (since $v_1 \neq 0$ as it is part of a basis of V). Then, for $k = 2, \dots, n$, we have $Tv_k = \mathcal{M}(T)_{1,k} v_1 + \dots + \mathcal{M}(T)_{k-1,k} v_{k-1} + \lambda_k v_k$ (since $\mathcal{M}(T)_{k,k} = \lambda_k$).
So, $Tv_k - \lambda_k v_k = (T - \lambda_k I)v_k \in \operatorname{span}(v_1, \dots, v_{k-1})$. Therefore, $T - \lambda_k I$ maps $\operatorname{span}(v_1, \dots, v_k)$ to $\operatorname{span}(v_1, \dots, v_{k-1})$. Thus,
$(T - \lambda_k I)|_{\operatorname{span}(v_1, \dots, v_k)} \in \mathcal{L}(\operatorname{span}(v_1, \dots, v_k))$ satisfies the following by the FToLM: $k = \dim \operatorname{span}(v_1, \dots, v_k) = \dim \operatorname{null}\left((T - \lambda_k I)|_{\operatorname{span}(v_1, \dots, v_k)}\right) + \dim \operatorname{range} \text{"}\ \text{"}$

$$k \leq \dim \operatorname{null}\left((T - \lambda_k I)|_{\operatorname{span}(v_1, \dots, v_k)}\right) + (k-1)$$

$$1 \leq \dim \operatorname{null}\left((T - \lambda_k I)|_{\operatorname{span}(v_1, \dots, v_k)}\right)$$

But this says $T - \lambda_k I$ has a nontrivial null space! That is, λ_k is an eigenvalue of T. Therefore, $\lambda_1, \dots, \lambda_n$ are all eigenvalues of T.
Now, since the poly. $q(z) = (z - \lambda_1) \cdots (z - \lambda_n)$ satisfies $q(T) = 0$ by the previous result. So, q is a poly. mult. of the min. poly. Since the zeros of the min. poly are the eigenvalues of T, this tells us that $\lambda_1, \dots, \lambda_n$ contains <u>all</u> the eigenvalues of T.

5.44 *necessary and sufficient condition to have an upper-triangular matrix*

Suppose V is finite-dimensional and $T \in \mathcal{L}(V)$. Then T has an upper-triangular matrix with respect to some basis of V if and only if the minimal polynomial of T equals $(z-\lambda_1)\cdots(z-\lambda_m)$ for some $\lambda_1, \ldots, \lambda_m \in \mathbf{F}$.

$\boxed{\text{Proof}}$ First, $\oint$ T has an U.T. matrix wrt some basis of V. Let $\alpha_1, \ldots, \alpha_n$ denote the entries on the diagonal of $\mathcal{M}(T)$. Define $q(z) = (z-\alpha_1)\cdots(z-\alpha_n)$. Then, $q(T) = 0$ by the previous results. Hence, q is a poly. mult. of the min. poly. of T. Therefore, the min poly is of the form $(z-\lambda_1)\cdots(z-\lambda_m)$ for some $\{\lambda_1, \ldots, \lambda_m\} \subseteq \{\alpha_1, \ldots, \alpha_n\}$.

For the other direction, $\oint$ the min. poly. of $T \in \mathcal{L}(V)$ is of the form $(z-\lambda_1)\cdots(z-\lambda_m)$ for some $\lambda_1, \ldots, \lambda_m \in \mathbb{F}$. We do induction on m!

Base Case: Let $m=1$. Then, $T - \lambda_1 I = 0$, & $T = \lambda_1 I$ & therefore $\mathcal{M}(T)$ is upper triangular w.r.t. any basis of V. (Since $\mathcal{M}(T) = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_1 \end{bmatrix}$).

Inductive Step: $\oint$ our result is true for all values less than $m > 1$. $\oint$ the min. poly. of T is $(z-\lambda_1)\cdots(z-\lambda_m)$ for some $\lambda_1, \ldots, \lambda_m \in \mathbb{F}$. Then, let $\boxed{U = \operatorname{range}(T - \lambda_m I)}$. Notice U is invariant under T since $T - \lambda_m I = p(T)$ for $p(z) = z - \lambda_m$. Hence $T|_U \in \mathcal{L}(U)$ is an operator & we can see that if $u \in U$, then $(T-\lambda_m I)v = u$ for some $v \in V$. So, $(T-\lambda_1 I)\cdots(T-\lambda_{m-1} I)u = (T-\lambda_1 I)\cdots(T-\lambda_{m-1}I)(T-\lambda_m I)v = 0v = 0$ & $(z-\lambda_1)\cdots(z-\lambda_{m-1})$ is a poly. mult. of the min. poly. of $T|_U$. So, the min poly. of $T|_U$ is of the form $(z-\beta_1)\cdots(z-\beta_t)$ for $\{\beta_1, \ldots, \beta_t\} \subseteq \{\lambda_1, \ldots, \lambda_{m-1}\}$. So, there is a basis, by our inductive hypothesis, such that $T|_U$ has an upper triangular matrix. Let $u_1, \ldots, u_M$ be this basis for U. Then, $Tu_k = T|_U u_k \in \operatorname{Span}(u_1, \ldots, u_k)$ for $k = 1, \ldots, M$. Extend $u_1, \ldots, u_M$ to a basis of V as $u_1, \ldots, u_M, v_1, \ldots, v_N$. For $k = 1, \ldots, N$, we have $Tv_k = Tv_k - \lambda_m v_k + \lambda_m v_k = (T - \lambda_m I)v_k + \lambda_m v_k$. By definition, $(T-\lambda_m I)v_k \in U$ & $Tv_k \in \operatorname{Span}(u_1, \ldots, u_M, v_1, \ldots, v_k)$. So, we are done!

5.47 *if* $\mathbf{F} = \mathbf{C}$, *then every operator on* V *has an upper-triangular matrix*

Suppose V is a finite-dimensional complex vector space and $T \in \mathcal{L}(V)$. Then T has an upper-triangular matrix with respect to some basis of V.