5.29 *$q(T) = 0 \iff q$ is a polynomial multiple of the minimal polynomial*

Suppose V is finite-dimensional, $T \in \mathcal{L}(V)$, and $q \in \mathcal{P}(\mathbf{F})$. Then $q(T) = 0$ if and only if q is a polynomial multiple of the minimal polynomial of T.

$\boxed{\text{Proof}}$ Suppose p is the minimal polynomial of T.
If $q \in \mathcal{P}(\mathbb{F})$ is s.t. $q(T) = 0$, then doing division, there are polynomials $s, r \in \mathcal{P}(\mathbb{F})$ s.t.

$$q = ps + r$$

w/ degree of r less than the degree of p.
Then, $0 = q(T) = p(T)s(T) + r(T) = 0\,s(T) + r(T) = r(T)$.
So, $r(T) = 0$ & $\deg r < \deg p$, so by minimality, r is the zero polynomial. That is, $q = ps$.
That is, q is a polynomial multiple of p.
Conversely, suppose $q = ps$ for some poly $s \in \mathcal{P}(\mathbb{F})$. Then,
$q(T) = p(T)s(T) = 0\,s(T) = 0$.

5.31 *minimal polynomial of a restriction operator*

Suppose V is finite-dimensional, $T \in \mathcal{L}(V)$, and U is a subspace of V that is invariant under T. Then the minimal polynomial of T is a polynomial multiple of the minimal polynomial of $T|_U$.

avoid this mess
~~$T: V \to V$~~
~~$T|_u : U \to V$~~
~~$\hat{T}|_u : U \to U$~~

$\boxed{\text{Proof}}$ Let p be the min. poly. of T.
Hence, $p(T) = 0$ & $p(T)v = 0v = 0$ for all $v \in V$.
So, $p(T)u = 0$ for all $u \in U$. Hence, $p(T|_u) = 0$ (where this is $0 \in \mathcal{L}(U)$). Thus, by the previous result, p is a poly. multiple of the min. poly. of $T|_u$.

5.32 *T not invertible $\iff$ constant term of minimal polynomial of T is 0*

Suppose V is finite-dimensional and $T \in \mathcal{L}(V)$. Then T is not invertible if and only if the constant term of the minimal polynomial of T is 0.

$\boxed{\text{Proof}}$ Let p be the mini. poly. of T.

T is not invertible $\iff$ 0 is an eigenvalue of T
$\iff$ 0 is a zero of p
$\iff$ $p(0)=0$
$\iff$ the constant term of p is zero.

Eigenvalues on Odd-Dimensional Real Vector Spaces

5.33 *even-dimensional null space*

Suppose $\mathbf{F} = \mathbf{R}$ and V is finite-dimensional. Suppose also that $T \in \mathcal{L}(V)$ and $b, c \in \mathbf{R}$ with $b^2 < 4c$. Then $\dim \operatorname{null}(T^2 + bT + cI)$ is an even number.

$\boxed{\text{Proof}}$ Recall that the null space of $p(T)$ is invariant under T for any polynomial $p \in \mathcal{P}(\mathbb{R})$. If necessary, replace V w/ $\operatorname{null}(T^2+bT+cI)$ & T with $T|_{\operatorname{null}(T^2+bT+cI)}$ so that we are only dealing w/ $\operatorname{null}(T^2+bT+cI)$. This lets us assume $T^2+bT+cI=0$ (since we are taking V to be $\operatorname{null}(T^2+bT+cI)$ if it is not already). We want to show $\dim(\operatorname{null}(T^2+bT+cI))=\dim V$ is even. $\oint$ $\lambda \in \mathbb{R}$ & $v \in V$ is s.t. $Tv=\lambda v$. Then,

$$0=(T^2+bT+cI)v=\lambda^2 v+b\lambda v+cv=\underbrace{\left((\lambda+\tfrac{b}{2})^2+c-\tfrac{b^2}{4}\right)}v$$

Since $b^2<4c$, $0<4c-b^2$ & $0<c-\frac{b^2}{4}$. So, $\swarrow$ is a positive #. Hence, $v=0$. So, T has no eigenvectors.

$\boxed{\text{proof}}$ (continued)

Let U be a subspace of V that is invariant under T. Moreover, $\ni$ $\dim U$ is as large as possible while remaining even. If $\underline{U=V}$, we are done. If not, let $w \in V$ be some vector s.t. $w \notin U$. Let $W = \operatorname{span}(w, Tw)$. Then, W is invariant under T, since $T(Tw) = T^2 w = -bTw - cw$ (via $T^2 + bT + cI = 0$, which means $T^2 = -bT - cI$). Moreover, since T has no eigenvectors, $Tw \neq \lambda w$ for any $\lambda \in \mathbb{F}$ $\ni$ $\dim W = 2$. Now, $\dim(U+W) = \dim U + \dim W - \dim(U \cap W) = \dim U + 2 - 0$, and $U+W$ is an even-dimensional subspace that is invariant under T, contradicting the maximality of $\dim U$. So, $U=V$ must have been the case!

5.34 *operators on odd-dimensional vector spaces have eigenvalues*

Every operator on an odd-dimensional real vector space has an eigenvalue.

$\boxed{\text{Proof}}$ We do induction.

Base case: & $\dim V = 1$. Then, if $T \in \mathcal{L}(V)$, every nonzero vector $v \in V$ is an eigenvector of T, since $Tv = \lambda v$ for some $\lambda \in \mathbb{R}$ (i.e. if $Tv = w$, then $Tw = \left(\frac{w}{v}\right)v$).

Induction step: S every operator on an odd dimensional vector space w/ $\dim V \leq n$ has an eigenvalue. We show this implies any operator on an odd dimensional vector space of dimension $\dim U = n+2$. Let p be the minimal poly. of $T \in \mathcal{L}(V)$. If p is divisible by $(x-\lambda)$, then $p(\lambda)=0$ & λ is an eigenvalue of T. So, if p is not divisible by $(x-\lambda)$ for any λ, then $p(x) = q(x)(x^2+bx+c)$ w/ $b^2 < 4c$ (Since $x^2+bx+c=0$ for $x = \frac{-b \pm \sqrt{b^2-4c}}{2}$ & $x \notin \mathbb{R}$ when $b^2<4c$).

So, $0 = p(T) = q(T)(T^2+bT+cI)$. Hence, $q(T)\big|_{\operatorname{range}(T^2+bT+cI)} = 0$

Since $\deg q < \deg p$ & p is the min. poly. of T, $\operatorname{range}(T^2+bT+cI) \neq U$.

By the FToLM, $\dim U = \dim \operatorname{null}(T^2+bT+cI) + \dim \operatorname{range}(T^2+bT+cI)$.

Since $\dim V$ is odd & $\dim \operatorname{null}(T^2+bT+cI)$ is even, $\dim \operatorname{range} T^2+bT+cI$ is odd. So, our inductive hypothesis applies to $T\big|_{\operatorname{range}(T^2+bT+cI)}$

Hence, $T\big|_{\operatorname{range}(T^2+bT+cI)}$ has an eigenvalue, and so then does T.