

5B The Minimal Polynomial

5.19 existence of eigenvalues

Every operator on a finite-dimensional nonzero complex vector space has an eigenvalue.

Proof $\$$ V is a finite-dim. vector space over $\mathbb{C}$
 $\$ T \in \mathcal{L}(V)$. Let $\dim V = n < \infty$. Choose any $0 \neq v \in V$.
Then, the list $v, Tv, T^2v, \dots, T^n v$ is linearly dependent,
since this is a list of length $n+1 > n$. This says there
exists coefficients $a_0, a_1, \dots, a_n \in \mathbb{C}$ s.t. not all are zero $\$$

$$0 = a_0 v + a_1 Tv + a_2 T^2 v + \dots + a_n T^n v$$

Now, this gives at least one polynomial (ex. $p(z) = a_0 + a_1 z + a_2 z^2 + \dots + a_n z^n$)
s.t. $p(T)v = 0$. Let us suppose that this polynomial is chosen
to be of minimal degree. So, by the fundamental theorem of algebra,
there exists a poly $q \in \mathcal{P}(\mathbb{C})$ $\$$ a value $\lambda \in \mathbb{C}$ s.t.
$$p(z) = (z - \lambda)q(z)$$

So, $0 = p(T)v = (T - \lambda I)q(T)v$. Since q has degree less than
 p , we know $q(T)v \neq 0$ (since we supposed $______$).
Hence, λ is an eigenvalue of T $\$$ $q(T)v$ is a corresponding
eigenvector.

Eigenvalues and the Minimal Polynomial

5.21 definition: monic polynomial

A monic polynomial is a polynomial whose highest-degree coefficient equals 1.

Ex $p(z) = 4z + 9z^2$ is not monic
 $q(z) = 5z + 11z^2 + z^3$ is monic

5.22 existence, uniqueness, and degree of minimal polynomial

Suppose V is finite-dimensional and $T \in \mathcal{L}(V)$. Then there is a unique monic polynomial $p \in \mathcal{P}(\mathbb{F})$ of smallest degree such that $p(T) = 0$. Furthermore, $\deg p \leq \dim V$.

Not proven yet!

Proof $\S \dim V = n < \infty \ \S T \in \mathcal{L}(V)$.

Write $\text{poly}(T) = \{p(T) : p \in \mathcal{P}(\mathbb{F})\}$. Now, since $0 \in \text{poly}(T)$ (i.e. $0 = 0(T)$) $\S q(T) + p(T) = (q+p)(T) \ \S \lambda(q(T)) = A_\ell(T)$, $\text{poly}(T) \subseteq \mathcal{L}(V)$. Since $\dim V < \infty$, $\dim \mathcal{L}(V) < \infty \ \S$ we can see that $\dim \text{poly}(T) = m < \infty$ as well. Hence, the list

$$I = T^0, T, T^2, \dots, T^m$$

is linearly dependent since it is of length $m+1$. Therefore, there exist $a_0, \dots, a_m$ not all 0 $\in \mathbb{F}$ st. $a_0 I + a_1 T + \dots + a_m T^m = 0$.

$\S$ that $k \leq m$ is the largest value st. $a_k \neq 0$ (i.e. $a_i = 0$ for $i > k$). Then, the polynomial $\frac{1}{a_k}(a_0 + a_1 z + \dots + a_k z^k) = p(z)$ is monic and satisfies $p(T) = \frac{1}{a_k}(a_0 I + a_1 T + \dots + a_k T^k + \dots + a_m T^m) = \frac{1}{a_k} 0 = 0$. So, such a poly exists. Hence, we can choose one of lowest degree.

Now, to show uniqueness, $\S p$ $\S q$ are monic polys of minimal degree st. $p(T) = q(T) = 0$. Now, $(p-q)(T) = 0$ and notice that if $p-q \neq 0$, then we have a poly of lower degree (since the coefficient on the highest degree term of both p $\S q$ is 1) which can be made monic by dividing by the coefficient of the highest power of z in it. this would contradict minimality, so $p=q$.

5.24 definition: *minimal polynomial*

Suppose V is finite-dimensional and $T \in \mathcal{L}(V)$. Then the minimal polynomial of T is the unique monic polynomial $p \in \mathcal{P}(F)$ of smallest degree such that $p(T) = 0$.

5.27 *eigenvalues are the zeros of the minimal polynomial*

Suppose V is finite-dimensional and $T \in \mathcal{L}(V)$.

- (a) The zeros of the minimal polynomial of T are the eigenvalues of T .
- (b) If V is a complex vector space, then the minimal polynomial of T has the form

$$(z - \lambda_1) \cdots (z - \lambda_m),$$

where $\lambda_1, \dots, \lambda_m$ is a list of all eigenvalues of T , possibly with repetitions.

Proof Let p be the minimal poly. of $T \in \mathcal{L}(V)$.
 Now, λ is a zero of p . Then, by the F.T.O.A.,
 $p(z) = (z - \lambda)q(z)$ where q is monic & of degree less than p . Hence, since $p(T) = 0$, $(T - \lambda I)q(T) = 0$.
 & b/c q is of lower degree than p , $q(T) \neq 0$. Hence, applying any $v \in V$ s.t. $q(T)v \neq 0$ (means this is possible), we get that $q(T)v \in \text{null}(T - \lambda I)$ & λ is an e.-val. of T .

Now, λ is an eigenvalue of T . Then, $\exists 0 \neq v \in V$ s.t. $Tv = \lambda v$. Accordingly, $T^k v = T^{k-1} \lambda v = \dots = \lambda^k v$. Hence, $p(T)v = p(\lambda)v$. Since p is the min. poly. of T , $p(T) = 0$, so $p(\lambda)v = 0$ & $p(\lambda) = 0$ since $v \neq 0$. Thus, λ is a zero of p .

For part b, just use part (a) & the F.T.O.A. to factor p .