$U^0 = \{\varphi \in V' : \varphi(u) = 0 \text{ for all } u \in U\}$

$T \in \mathcal{L}(V, W)$
$T' \in \mathcal{L}(W', V')$

$$\begin{tikzcd} V \arrow[r, "T"] \arrow[dr, dashed, "T'(\varphi)"'] & W \arrow[d, "\varphi \in W'"] \\ & \mathbb{F} \end{tikzcd}$$

$T'(\varphi) = \varphi \circ T$

3.125 *dimension of the annihilator*

Suppose V is finite-dimensional and U is a subspace of V. Then

$$\dim U^0 = \dim V - \dim U.$$

$\boxed{\text{proof}}$ Define $i \in \mathcal{L}(U, V)$ by $i(u) = u$. So, $i' \in \mathcal{L}(V', U')$

By the FToLM, $\dim \operatorname{range} i' + \dim \operatorname{null} i' = \dim V'$. But, notice: $\operatorname{null} i' = \{\varphi \in V' : i'(\varphi) = 0\} = \{\varphi \in V' : \varphi \circ i = 0\} = \{\varphi \in V' : \varphi(u) = 0 \ \forall u \in U\}$

So, $\operatorname{null} i' = U^0$. Since we have shown $\dim V = \dim V'$ when V is finite-dimensional, we have $\dim \operatorname{range} i' + \dim U^0 = \dim V$.

Hence, if $\dim \operatorname{range} i' = \dim U$, we are done!

notice: $\operatorname{range} i' = \{\psi \in U' : \exists \varphi \in V' \ i'(\varphi) = \psi\} = \{\psi \in U' : \exists \varphi \in V' \ \varphi(u) = \psi(u) \ \forall u \in U\}$

Since every $\psi \in U'$ can be extended to a linear functional on V (we proved this in the homework), this says $\operatorname{range} i' = U'$ (i.e. i' is surjective).

But, $\dim U' = \dim U$, so we are finished.

3.127 *condition for the annihilator to equal* $\{0\}$ *or the whole space*

Suppose V is finite-dimensional and U is a subspace of V. Then

(a) $U^0 = \{0\} \iff U = V$;

(b) $U^0 = V' \iff U = \{0\}$.

$\boxed{\text{proof}}$ For part (a):

$$\begin{aligned} U^0 = \{0\} &\iff \dim U^0 = 0 \\ &\iff \dim V - \dim U = 0 \\ &\iff \dim V = \dim U \\ &\iff V = U \end{aligned}$$

For part (b):

$$\begin{aligned} U^0 = V' &\iff \dim U^0 = \dim V' \\ &\iff \dim V - \dim U = \dim V \\ &\iff \dim U = 0 \\ &\iff U = \{0\}. \end{aligned}$$

3.128 *the null space of T'*

Suppose V and W are finite-dimensional and $T \in \mathcal{L}(V, W)$. Then

(a) $\operatorname{null} T' = (\operatorname{range} T)^0$; ✓

(b) $\dim \operatorname{null} T' = \dim \operatorname{null} T + \dim W - \dim V$.

$\boxed{\text{proof}}$ φ $\varphi \in \operatorname{null} T'$. So, $0 = T'(\varphi) = \varphi \circ T$. Thus, for every $v \in V$, $\varphi(T(v)) = 0$. Hence, $\varphi \in (\operatorname{range} T)^0$. So, $\operatorname{null} T' \subseteq (\operatorname{range} T)^0$.

Now, φ $\psi \in (\operatorname{range} T)^0$. Then, $\psi(T(v)) = 0$ for every $v \in V$, i.e. $(T'(\psi))(v) = 0$ for all $v \in V$. So, $\psi \in \operatorname{null} T'$. Hence, $\operatorname{null} T' \supseteq (\operatorname{range} T)^0$.

Therefore, $\operatorname{null} T' = (\operatorname{range} T)^0$.

For the second part, observe:

$$
\begin{aligned}
\dim \operatorname{null} T' &= \dim (\operatorname{range} T)^0 \\
&= \dim W - \dim \operatorname{range} T \\
&= \dim W - (\dim V - \dim \operatorname{null} T) \\
&= \dim \operatorname{null} T + \dim W - \dim V.
\end{aligned}
$$

Since $\dim V = \dim \operatorname{null} T + \dim \operatorname{range} T$

3.129 *T surjective is equivalent to T' injective*

Suppose V and W are finite-dimensional and $T \in \mathcal{L}(V, W)$. Then

$$T \text{ is surjective} \iff T' \text{ is injective.}$$

$\boxed{\text{proof}}$ If $T \in \mathcal{L}(V, W)$ is surjective, then $\operatorname{range} T = W$.

But:

$$
\begin{aligned}
\operatorname{range} T = W &\iff (\operatorname{range} T)^0 = \{0\} \\
&\iff \operatorname{null} T' = \{0\} \\
&\iff T' \text{ is injective}
\end{aligned}
$$

So, we are done!

3.130 *the range of T'*

Suppose V and W are finite-dimensional and $T \in \mathcal{L}(V, W)$. Then

(a) $\dim \operatorname{range} T' = \dim \operatorname{range} T$;

(b) $\operatorname{range} T' = (\operatorname{null} T)^0$.

$\boxed{\text{proof}}$ For part (a), we have $\dim \operatorname{range} T' = \dim W' - \dim \operatorname{null} T'$

$$= \dim W - \dim (\operatorname{range} T)^0$$
$$= \cancel{\dim W} - (\cancel{\dim W} - \dim \operatorname{range} T)$$
$$= \dim \operatorname{range} T$$

For part (b), $\varphi \in \operatorname{range} T'$. Then, $\exists \psi \in W'$ s.t. $T'(\psi) = \varphi$. Now, if $v \in \operatorname{null} T$, then $\varphi(v) = T'(\psi)(v) = \psi(T(v)) = \psi(0) = 0$. Hence, $\varphi \in (\operatorname{null} T)^0$. So, $\operatorname{range} T' \subseteq (\operatorname{null} T)^0$.

To show $\operatorname{range} T' = (\operatorname{null} T)^0$, we show they have the same dimension.

Notice: $\dim \operatorname{range} T' = \dim \operatorname{range} T = \dim V - \dim \operatorname{null} T = \dim (\operatorname{null} T)^0$.

So, $\operatorname{range} T' = (\operatorname{null} T)^0$.

3.131 *T injective is equivalent to T' surjective*

Suppose V and W are finite-dimensional and $T \in \mathcal{L}(V, W)$. Then

$$T \text{ is injective} \iff T' \text{ is surjective.}$$

$\boxed{\text{proof}}$ T is injective $\iff \operatorname{null} T = \{0\}$

$$\iff (\operatorname{null} T)^0 = V'$$
$$\iff \operatorname{range} T' = V'$$
$$\iff T' \text{ is surjective.}$$

Matrix of Dual of Linear Map

3.132 *matrix of T' is transpose of matrix of T*

Suppose V and W are finite-dimensional and $T \in \mathcal{L}(V, W)$. Then

$$\mathcal{M}(T') = (\mathcal{M}(T))^t.$$

$\boxed{\text{Proof}}$ If $\psi_1, \dots, \psi_m$ is a basis for W' & $\varphi_1, \dots, \varphi_n$ is a basis for W, where $\psi_1, \dots, \psi_m$ is the dual basis of $w_1, \dots, w_m$ & $\varphi_1, \dots, \varphi_n$ is the dual basis of $v_1, \dots, v_n$, then we have

$$T'(\psi_j) = \sum_{r=1}^{n} \mathcal{M}(T')_{r,j} \varphi_r$$

Now, apply this to v_k.

$$\psi_j(Tv_k) = T'(\psi_j)(v_k) = \sum_{r=1}^{n} \mathcal{M}(T')_{r,j} \varphi_r(v_k) = \mathcal{M}(T')_{k,j}$$

We also know $\psi_j(T(v_k)) = \psi_j\left(\sum_{r=1}^{m} \mathcal{M}(T)_{r,k} w_r\right)$

$$= \sum_{r=1}^{m} \mathcal{M}(T)_{r,k} \psi_j(w_r)$$

$$= \mathcal{M}(T)_{j,k}$$

So, $\mathcal{M}(T')_{k,j} = \mathcal{M}(T)_{j,k}$ and

$$\mathcal{M}(T') = (\mathcal{M}(T))^T.$$

3.133 *column rank equals row rank*

Suppose $A \in \mathbf{F}^{m,n}$. Then the column rank of A equals the row rank of A.