

3F Duality

Dual Space and Dual Map

3.108 definition: linear functional

A linear functional on V is a linear map from V to $\mathbb{F}$. In other words, a linear functional is an element of $\mathcal{L}(V, \mathbb{F})$.

3.110 definition: dual space, V'

The dual space of V , denoted by V' , is the vector space of all linear functionals on V . In other words, $V' = \mathcal{L}(V, \mathbb{F})$.

3.111 $\dim V' = \dim V$

Suppose V is finite-dimensional. Then V' is also finite-dimensional and

$$\dim V' = \dim V.$$

Proof

We know that if V & W are finite-dim vector spaces, then $\dim \mathcal{L}(V, W) = (\dim V)(\dim W)$.

So, if $W = \mathbb{F}$, $\dim W = \dim \mathbb{F} = 1$ and we get

$$\underline{\dim V'} = \dim \mathcal{L}(V, \mathbb{F}) = (\dim V)(\dim \mathbb{F}) = \underline{\dim V}.$$

3.112 definition: dual basis

If $\underline{v_1, \dots, v_n}$ is a basis of V , then the dual basis of $\underline{v_1, \dots, v_n}$ is the list $\underline{\varphi_1, \dots, \varphi_n}$ of elements of V' , where each $\underline{\varphi_j}$ is the linear functional on V such that

$$\underline{\varphi_j(v_k)} = \begin{cases} 1 & \text{if } k = j, \\ 0 & \text{if } k \neq j. \end{cases}$$

Since φ_j is defined on a basis, it is defined everywhere by linearity — i.e. $\varphi_j(v) = \varphi_j(a_1 v_1 + \dots + a_n v_n) = a_1 \varphi_j(v_1) + \dots + a_n \varphi_j(v_n) = a_j \varphi_j(v_j) = \underline{a_j}$.

3.114 dual basis gives coefficients for linear combination

Suppose $\underline{v}_1, \dots, \underline{v}_n$ is a basis of $\underline{V}$ and $\underline{\varphi}_1, \dots, \underline{\varphi}_n$ is the dual basis. Then

$$\underline{v} = \underline{\varphi}_1(\underline{v})\underline{v}_1 + \dots + \underline{\varphi}_n(\underline{v})\underline{v}_n$$

for each $\underline{v} \in \underline{V}$.

Proof Since $v_1, \dots, v_n$ is a basis of V , $\exists a_1, \dots, a_n \in \mathbb{F}$ st.

$v = a_1 v_1 + \dots + a_n v_n$. So, by linearity, we get

$$\varphi_j(v) = \varphi_j(a_1 v_1 + \dots + a_n v_n) = a_1 \varphi_j(v_1) + \dots + a_n \varphi_j(v_n) = a_j \varphi_j(v_j) = a_j$$

for each $j \in \{1, \dots, n\}$. So, this says

$$v = a_1 v_1 + \dots + a_n v_n = \varphi_1(v) v_1 + \dots + \varphi_n(v) v_n.$$

3.116 dual basis is a basis of the dual space

Suppose $\underline{V}$ is finite-dimensional. Then the dual basis of a basis of V is a basis of $\underline{V}'$.

Proof $\S$ $v_1, \dots, v_n$ is a basis of V . Let $\varphi_1, \dots, \varphi_n$ denote the corresponding dual basis. For lin. indep., $\S$ $a_1, \dots, a_n \in \mathbb{F}$ are

st. $a_1 \varphi_1 + \dots + a_n \varphi_n = 0$ (where $0 \in \mathcal{L}(V, \mathbb{F}) = V'$ is the map $0(v) = 0 \forall v \in V$).

But then, $0 = 0(v_j) = (a_1 \varphi_1 + \dots + a_n \varphi_n)(v_j) = a_1 \varphi_1(v_j) + \dots + a_n \varphi_n(v_j) = a_j \varphi_j(v_j) = a_j$.

So, since this holds for all $j \in \{1, \dots, n\}$, this says $a_1 = \dots = a_n = 0$.

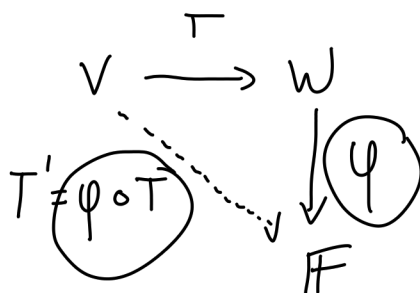
Thus, $\varphi_1, \dots, \varphi_n$ is a lin. indep. list in V' of length $\dim V'$, so it is a basis.

3.118 definition: dual map, T'

Suppose $T \in \mathcal{L}(V, W)$. The dual map of T is the linear map $T' \in \mathcal{L}(W', V')$ defined for each $\varphi \in W'$ by

$$T'(\varphi) = \varphi \circ T.$$

Visualize



the claimed linearity is a consequence of the definition of T' : it is just composition of two linear maps!

Notice: T' basically "flops" the domain & codomain of T around. Technically, it flops & dualizes, but since $V \cong V'$ (also, $W \cong W'$) are isomorphic if finite dim., this is just relabeling.

$$\begin{aligned} T'(\varphi + \psi) &= (\varphi + \psi) \circ T \\ &= \varphi \circ T + \psi \circ T \\ &= T'(\varphi) + T'(\psi) \end{aligned}$$

3.120 algebraic properties of dual maps

Suppose $T \in \mathcal{L}(V, W)$. Then

- (a) $(S + T)' = S' + T'$ for all $S \in \mathcal{L}(V, W)$;
- (b) $(\lambda T)' = \lambda T'$ for all $\lambda \in F$;
- (c) $(ST)' = T' S'$ for all $S \in \mathcal{L}(W, U)$.

$$\begin{aligned} T'(\lambda \varphi) &= \lambda \varphi \circ T \\ &= \lambda (\varphi \circ T) \\ &= \lambda T'(\varphi) \end{aligned}$$

Proof

(a) Let $\varphi \in W'$. Then, $(S + T)'(\varphi) = \varphi \circ (S + T) = \varphi \circ S + \varphi \circ T = S'(\varphi) + T'(\varphi)$

(b) Let $\varphi \in W'$ & $\lambda \in F$. Then, $(\lambda T)'(\varphi) = \varphi \circ (\lambda T) = \lambda (\varphi \circ T) = \lambda T'(\varphi)$
Since φ is linear

(c) & $S \in \mathcal{L}(W, U)$ & $\varphi \in U'$. Then, $(ST)'(\varphi) = \varphi \circ ST = (\varphi \circ S) \circ T = T'(\varphi \circ S) = T'(S'(\varphi)) = (T'S')(\varphi)$.

Null Space and Range of Dual of Linear Map

3.121 definition: *annihilator*, U^0

For $\underline{U} \subseteq V$, the *annihilator* of $\underline{U}$, denoted by $\underline{U}^0$, is defined by

$$\underline{U}^0 = \{ \underline{\varphi} \in V' : \underline{\varphi}(\underline{u}) = 0 \text{ for all } \underline{u} \in \underline{U} \}.$$

3.124 the annihilator is a subspace

Suppose $\underline{U} \subseteq V$. Then $\underline{U}^0$ is a subspace of $\underline{V}'$.

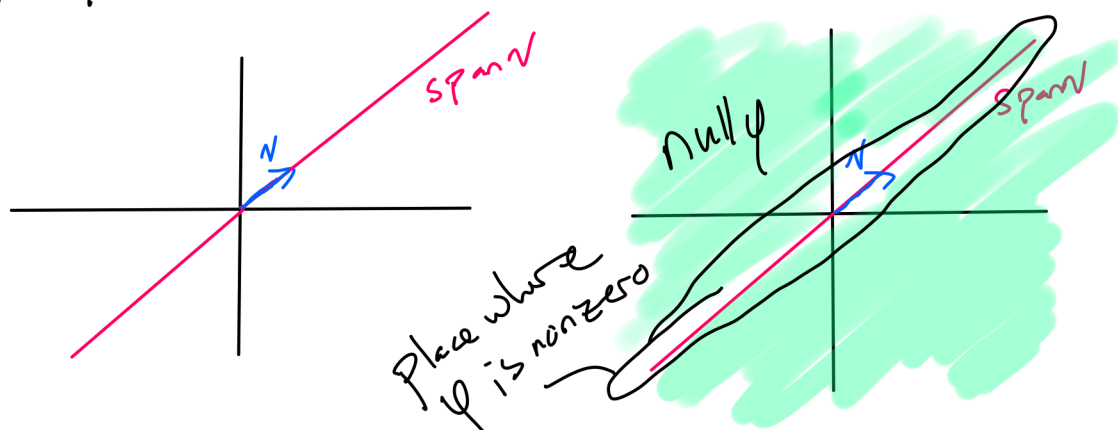
Proof We need only show $0 \in U^0$, U^0 is closed under addition, & U^0 is closed under scalar multiplication.

First, note that $0 \in V' = \mathcal{L}(V, \mathbb{F})$ is defined by $0(v) = 0 \forall v \in V$. So, $0(u) = 0$ for all $u \in U$ & $0 \in U^0$.

Now, & $\varphi, \psi \in U^0$. Then, $\varphi(u) = 0, \psi(u) = 0 \forall u \in U$. Hence, we have $(\varphi + \psi)(u) = \varphi(u) + \psi(u) = 0 + 0 = 0$, so $\varphi + \psi \in U^0$ as well.

Finally, & $\varphi \in U^0$ & $\lambda \in \mathbb{F}$. Then, $(\lambda\varphi)(u) = \lambda(\varphi(u)) = \lambda(0) = 0 \forall u \in U$, so $\lambda\varphi \in U^0$ as well.

Idea We can look at vectors & the subspaces they span. The idea of a dual space is to invert this:



The idea is to find something in V' that is 0 except on $\text{span } v$