3E Products and Quotients of Vector Spaces

Products of Vector Spaces

3.87 definition: *product of vector spaces*

Suppose $V_1, \dots, V_m$ are vector spaces over **F**. ← Same field!

- The *product* $V_1 \times \cdots \times V_m$ is defined by

$$V_1 \times \cdots \times V_m = \{(v_1, \dots, v_m) : v_1 \in V_1, \dots, v_m \in V_m\}.$$

- Addition on $V_1 \times \cdots \times V_m$ is defined by

$$(u_1, \dots, u_m) + (v_1, \dots, v_m) = (u_1 + v_1, \dots, u_m + v_m).$$

- Scalar multiplication on $V_1 \times \cdots \times V_m$ is defined by

$$\lambda(v_1, \dots, v_m) = (\lambda v_1, \dots, \lambda v_m).$$

3.89 *product of vector spaces is a vector space*

Suppose $V_1, \dots, V_m$ are vector spaces over **F**. Then $V_1 \times \cdots \times V_m$ is a vector space over **F**.

3.92 *dimension of a product is the sum of dimensions*

Suppose $V_1, \dots, V_m$ are finite-dimensional vector spaces. Then $V_1 \times \cdots \times V_m$ is finite-dimensional and

$$\boxed{\dim(V_1 \times \cdots \times V_m)} = \dim V_1 + \cdots + \dim V_m.$$

Proof Let's construct a basis for $V_1 \times \cdots \times V_m$.

Let $v_{1,k}, \dots, v_{n_k,k}$ be a basis for V_k where $\dim V_k = n_k$.

We take $\underbrace{(v_{1,1}, 0, \dots, 0), \dots, (v_{n_1,1}, 0, \dots, 0)}_{\dim V_1}, \underbrace{(0, v_{1,2}, 0, \dots, 0), \dots, (0, v_{n_2,2}, 0, \dots, 0)}_{\dim V_2}, \dots, (0, \dots, 0, v_{n_m,m})$ & so on

(i.e. take the list of vectors in $V_1 \times \cdots \times V_m$ where everything is 0 except for one component, which is a vector in a basis of the corresponding space).

Since $v_k \in V_k$ can be written uniquely as $v_k = a_1 v_{1,k} + \dots + a_{n_k} v_{n_k,k}$, then $(0, \dots, 0, v_k, 0, \dots, 0) \in V_1 \times \cdots \times V_m$ can be written uniquely as a linear combo of

We can write every $(v_1, \dots, v_m) \in V_1 \times \cdots \times V_m$ uniquely as a linear combo of by breaking apart $(v_1, \dots, v_m) = (v_1, 0, \dots, 0) + \dots + (0, 0, \dots, 0, v_m)$. Hence, $\dim V_1 \times \cdots \times V_m$ is the sum of the dimensions of the V_i's.

3.93 *products and direct sums*

Suppose that $V_1, \dots, V_m$ are subspaces of V. Define a linear map $\Gamma : V_1 \times \cdots \times V_m \to V_1 + \cdots + V_m$ by

$$\Gamma(v_1, \dots, v_m) = v_1 + \cdots + v_m.$$

Then $V_1 + \cdots + V_m$ is a direct sum if and only if Γ is injective.

$\boxed{\text{Proof}}$ If Γ is injective, then $\Gamma(v_1, \dots, v_m) = 0 \Rightarrow (v_1, \dots, v_m) = 0$. So, the only way to write 0 as a sum of the form $v_1 + \dots + v_m = 0$ with $v_k \in V_k$ is $v_1 = \dots = v_m = 0$, which says $V_1 + \dots + V_m$ is direct. Now, if $V_1 \oplus \dots \oplus V_m$, then Γ must be injective, since directness implies that $v_1 + \dots + v_m = 0$ for $v_k \in V_k$ means $v_1 = \dots = v_m = 0$, so null $\Gamma = \{0\}$ & Γ is injective.

3.94 *a sum is a direct sum if and only if dimensions add up*

Suppose V is finite-dimensional and $V_1, \dots, V_m$ are subspaces of V. Then $V_1 + \cdots + V_m$ is a direct sum if and only if

$$\dim(V_1 + \cdots + V_m) = \dim V_1 + \cdots + \dim V_m.$$

← Notice: no products!

$\boxed{\text{Proof}}$ First, note that Γ (from the previous theorem) is surjective. By the FToLM, Γ is injective if & only if $\dim(V_1 + \dots + V_m) = \dim(V_1 \times \dots \times V_m)$ (since $\dim V_1 \times \dots V_m = \dim \operatorname{null} \Gamma + \dim \operatorname{range} \Gamma = \dim \operatorname{null} \Gamma + \dim(V_1 + \dots + V_m)$).

Now, by the previous result, Γ is injective if & only if $V_1 + \dots + V_m$ is direct. So, since $\dim(V_1 \times \dots \times V_m) = \dim V_1 + \dots + \dim V_m$, we have $\dim(V_1 + \dots V_m) = \dim V_1 + \dots + \dim V_m$ if & only if this sum is direct.

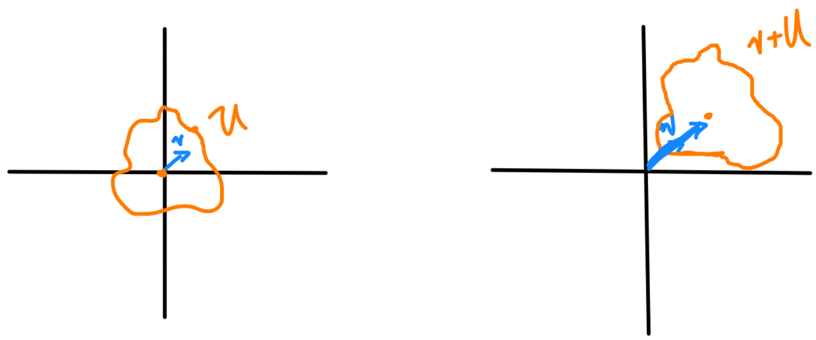
Quotient Spaces

3.95 notation: $v + U$

Suppose $v \in V$ and $U \subseteq V$. Then $v + U$ is the subset of V defined by

$$v + U = \{v + u : u \in U\}.$$

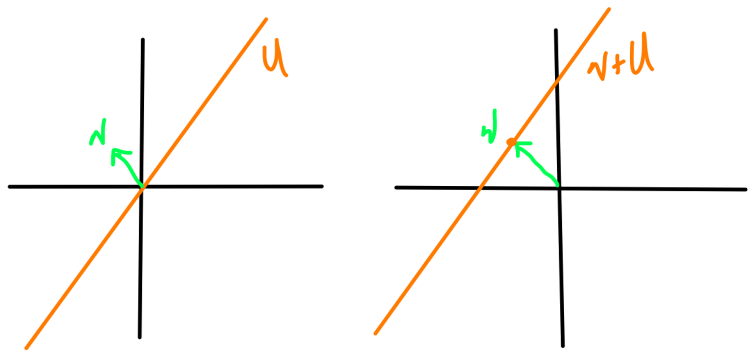
Visualize



3.97 definition: *translate*

For $v \in V$ and U a subset of V, the set $v + U$ is said to be a *translate* of U.

Visualize

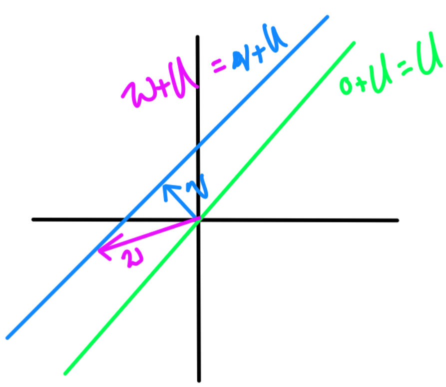


3.99 definition: *quotient space,* V/U

Suppose U is a subspace of V. Then the *quotient space* V/U is the set of all translates of U. Thus

$$V/U = \{v + U : v \in V\}.$$

Visualize



3.101 *two translates of a subspace are equal or disjoint*

Suppose U is a subspace of V and $v, w \in V$. Then

$$v - w \in U \iff v + U = w + U \iff (v + U) \cap (w + U) \neq \emptyset.$$

$\boxed{\text{Proof}}$ Suppose $v-w \in U$. If $u \in U$, then observe that

$v+U \ni v+u = v+u+w-w = w+((v-w)+u) \in w+U$. So, $v+U \subseteq w+U$.

Likewise, repeating the argument w/ v & w interchanged, $w+U \subseteq v+U$.

So, $v-w \in U \Longrightarrow v+U = w+U$.

Now, suppose $v+U = w+U$. Then, $(v+U) \cap (w+U) = v+U = w+U \neq \emptyset$.

So, $v+U = w+U \Rightarrow (v+U) \cap (w+U) \neq \emptyset$.

Finally, suppose $(v+U) \cap (w+U) \neq \emptyset$. Then, $\exists\, u_1, u_2 \in U$ s.t.

$v+u_1 = w+u_2$. Rearranging, this says $v-w = u_2 - u_1 \in U$.

So, $(v+U) \cap (w+U) \neq \emptyset \Rightarrow v-w \in U$.

3.102 definition: *addition and scalar multiplication on V/U*

Suppose U is a subspace of V. Then *addition* and *scalar multiplication* are defined on V/U by

$$(v + U) + (w + U) = (v + w) + U$$
$$\lambda(v + U) = (\lambda v) + U$$

for all $v, w \in V$ and all $\lambda \in \mathbf{F}$.

3.103 *quotient space is a vector space*

Suppose U is a subspace of V. Then V/U, with the operations of addition and scalar multiplication as defined above, is a vector space.

$\boxed{\text{Proof}}$ The problem we want to show does not happen is the following: If $v_1 + U = v_2 + U$ & $w_1 + U = w_2 + U$, then could we get $(v_1 + w_1) + U \neq (v_2 + w_2) + U$?

We show this never happens. Suppose $v_1 + U = v_2 + U$ & $w_1 + U = w_2 + U$. Then, $v_1 - v_2 \in U$ & $w_1 - w_2 \in U$. Now, this says, since U is a subspace, that $(v_1 - v_2) + (w_1 - w_2) \in U$. So, this says $(v_1 + w_1) - (v_2 + w_2) \in U$. But, we just showed that this implies $(v_1 + w_1) + U = (v_2 + w_2) + U$. So, addition makes sense!

Likewise, suppose $\lambda \in \mathbb{F}$ & $v_1 + U = v_2 + U$. Since U is a subspace, it closed under scaling, so $v_1 - v_2 \in U$ implies $\lambda(v_1 - v_2) \in U$. Hence, $\lambda v_1 - \lambda v_2 \in U$ & $\lambda v_1 + U = \lambda v_2 + U$. So, scalar mult. makes sense too!

With this, the rest is routine (exercise).

3.104 definition: *quotient map, π*

Suppose U is a subspace of V. The *quotient map* $\pi: V \to V/U$ is the linear map defined by

$$\pi(v) = v + U$$

for each $v \in V$.

3.105 *dimension of quotient space*

Suppose V is finite-dimensional and U is a subspace of V. Then

$$\dim V/U = \dim V - \dim U.$$

$\boxed{\text{Proof}}$ Let π be the quotient map. Then, since $\pi(v) = v + U$ & $V/U = \{v + U : v \in V\}$, π is surjective.

Now, notice that $\pi(v) = 0 + U$ (i.e. $v \in \operatorname{null} \pi$), then $v + U = 0 + U$, which means $v = v - 0 \in U$. So, $\operatorname{null} \pi = U$.

By the FToLM,
$$\dim V = \dim \operatorname{range} \pi + \dim \operatorname{null} \pi$$
$$\dim V = \dim V/U + \dim U$$
$$\Rightarrow \dim V/U = \dim V - \dim U.$$

3.106 notation: $\widetilde{T}$

Suppose $T \in \mathcal{L}(V, W)$. Define $\widetilde{T}: V/(\operatorname{null} T) \to W$ by

$$\widetilde{T}(v + \operatorname{null} T) = Tv.$$

This makes sense: If $v + \operatorname{null} T = u + \operatorname{null} T$, then $v - u \in \operatorname{null} T$ & $T(v-u) = 0$, i.e. $Tv = Tu$.

So, if $v + \operatorname{null} T = u + \operatorname{null} T$, $\widetilde{T}(v + \operatorname{null} T) = Tv = Tu = \widetilde{T}(u + \operatorname{null} T)$

that is, $\widetilde{T}$ does not depend on the choice of v used to represent $v + \operatorname{null} T$.

3.107 ***null space and range of $\widetilde{T}$***

Suppose $T \in \mathcal{L}(V, W)$. Then

(a) $\widetilde{T} \circ \pi = T$, where π is the quotient map of V onto $V/(\operatorname{null} T)$; $\pi : V \to V/\operatorname{null} T$

(b) $\widetilde{T}$ is injective;

(c) $\operatorname{range} \widetilde{T} = \operatorname{range} T$;

(d) $V/(\operatorname{null} T)$ and $\operatorname{range} T$ are isomorphic vector spaces.

$\boxed{\text{Proof}}$

(a) If $v \in V$, then $(\widetilde{T} \circ \pi)(v) = \widetilde{T}(\pi(v)) = \widetilde{T}(v + \operatorname{null} T) = Tv$

So, $\widetilde{T} \circ \pi = T$

(b) We show injectivity by showing $\operatorname{null} \widetilde{T}$ is trivial (i.e. $\operatorname{null} \widetilde{T} = \{0 + \operatorname{null} T\}$). & $\widetilde{T}(v + \operatorname{null} T) = 0$. Then, $Tv = 0$, since $\widetilde{T}(v + \operatorname{null} T) = Tv$. So, $v \in \operatorname{null} T$ & $v + \operatorname{null} T = 0 + \operatorname{null} T$ since $v - 0 \in \operatorname{null} T$. Hence, $\widetilde{T}$ is inj.

(c) If $w \in \operatorname{range} T$, then $\exists v \in V$ s.t. $Tv = w$. So, $\widetilde{T}(v + \operatorname{null} T) = Tv = w$ & $w \in \operatorname{range} \widetilde{T}$ as well.

Write this argument in reverse to get $w \in \operatorname{range} T$ whenever $w \in \operatorname{range} \widetilde{T}$.

(d) From (b) & (c), the map $R : V/\operatorname{null} T \to \operatorname{range} T$ defined by $R(v + \operatorname{null} T) = Tv$ is the same as $\widetilde{T}$ but with a different codomain. So, R is inj. & surject.