3D Invertibility and Isomorphisms

Invertible Linear Maps

3.59 definition: *invertible, inverse*

- A linear map $T \in \mathcal{L}(V, W)$ is called *invertible* if there exists a linear map $S \in \mathcal{L}(W, V)$ such that ST equals the identity operator on V and TS equals the identity operator on W.
- A linear map $S \in \mathcal{L}(W, V)$ satisfying $ST = I_V$ and $TS = I_W$ is called an *inverse* of T (note that the first I is the identity operator on V and the second I is the identity operator on W).

3.60 *inverse is unique*

An invertible linear map has a unique inverse.

$\boxed{\text{Proof}}$ § $T \in \mathcal{L}(V, W)$ is invertible. Then, $\exists S \in \mathcal{L}(W, V)$ s.t. $TS = I_W$ & $ST = I_V$. Suppose $S' \in \mathcal{L}(W, V)$ satisfies $TS' = I_W$ & $S'T = I_V$. Then, we have

$$S = SI_W = S(TS') = (ST)S' = I_V S' = S'$$

So, $S = S'$ & inverses are unique!

3.61 notation: T^{-1}

If T is invertible, then its inverse is denoted by T^{-1}. In other words, if $T \in \mathcal{L}(V, W)$ is invertible, then T^{-1} is the unique element of $\mathcal{L}(W, V)$ such that $T^{-1}T = I_V$ and $TT^{-1} = I_W$.

3.63 *invertibility $\iff$ injectivity and surjectivity*

A linear map is invertible if and only if it is injective and surjective.

$\boxed{\text{Proof}}$ First, assume $T \in \mathcal{L}(V,W)$ is invertible. Then, observe that if $w \in W$, then $T(T^{-1}w) = Iw = w$. So, T is surjective. For injectivity, $\oint$ $u, v \in V$ are s.t. $Tu = Tv$. Then, $u = T^{-1}(Tu) = T^{-1}(Tv) = v$, so T is injective.

- -

Now, $\oint$ $T \in \mathcal{L}(V,W)$ is injective $\&$ surjective. For $w \in W$, define $\underline{Sw}$ to be the <u>unique element of V</u> s.t. $\underline{TSw = w}$. This can be done because: T is surjective, so $\exists v \in V$ s.t. $Tv = w$ for whatever w we chose, and, since T is injective, if $Tv = w = Tu$, then $v = u$ (so, this choice is unique $\&$ S is well defined). By construction, $TS = I_W$. Now, observe that $T((ST)(v)) = (TS)(Tv) = Tv$. Since T is injective, $T(STv) = Tv$ implies $STv = v$. So, $ST = I_V$. All that remains is to show $S \in \mathcal{L}(W,V)$. $\oint$ $w_1, w_2 \in W$. Then, $T(Sw_1 + Sw_2) = TSw_1 + TSw_2 = w_1 + w_2$. Thus, $Sw_1 + Sw_2$ is the unique element that T maps to $w_1 + w_2$. By the definition of S, this says $S(w_1 + w_2) = Sw_1 + Sw_2$. Similarly, let $\lambda \in \mathbb{F}$. Then, $T(\lambda Sw) = \lambda TSw = \lambda w$, and $S(\lambda w) = \lambda Sw$ by the definition of S. So, $S \in \mathcal{L}(W,V)$

That is, T is invertible $\&$ $S = T^{-1}$.

3.65 *injectivity is equivalent to surjectivity* (*if* $\dim V = \dim W < \infty$)

Suppose that V and W are finite-dimensional vector spaces, $\dim V = \dim W$, and $T \in \mathcal{L}(V, W)$. Then

$$T \text{ is invertible} \iff T \text{ is injective} \iff T \text{ is surjective.}$$

$\boxed{\text{Proof}}$ By the FToLM, we have

$$\dim V = \dim \operatorname{null} T + \dim \operatorname{range} T$$

Now, if T is injective, $\operatorname{null} T = \{0\}$ & this says $\dim V = \dim \operatorname{range} T$ and, since $\dim V = \dim W$, $\dim \operatorname{range} T = \dim W$. Hence, $\operatorname{range} T = W$, so T is surjective as well. By the previous result, it is also invertible.

Likewise, if T is surjective, $\operatorname{range} T = W$ & we have $0 = \dim V - \dim \operatorname{range} T = \dim \operatorname{null} T$. So, $\operatorname{null} T = \{0\}$ & T is injective. By the previous result, it is also invertible.

$$\begin{tikzcd}
T \text{ injective} \arrow[rr, Leftrightarrow] \arrow[dr, Leftrightarrow] & & T \text{ surjective} \arrow[dl, Leftrightarrow] \\
& T \text{ invertible} &
\end{tikzcd}$$

$\boxed{\text{Ex}}$ Define $T \in \mathcal{L}(\mathcal{P}(\mathbb{R}))$ by $(Tp)(x) = ((x^2+5x+7)p(x))''$.

We will show that for any $q \in \mathcal{P}(\mathbb{R})$, there is some solution $p \in \mathcal{P}(\mathbb{R})$ to $Tp = q$. Let $m = \deg q$. Let $T' \in \mathcal{L}(\mathcal{P}_m(\mathbb{R}))$ be given by $(T'p)(x) = ((x^2+5x+7)p(x))''$.

Notice that if a polynomial has a second derivative of zero, then it is a poly of degree at most 1. Since multiplying a degree t poly by x^2+5x+7 produces a degree $t+2$ poly, $T'p = 0$ only when $p = 0$. So, T' is injective. Hence, invertible. So, $p = (T')^{-1}q$ & since T & T' agree, $T(T')^{-1}q = q$ as well.

3.68 $ST = I \iff TS = I$ *(on vector spaces of the same dimension)*

Suppose V and W are finite-dimensional vector spaces ~~of the same dimension~~, $S \in \mathcal{L}(V, W)$, and $T \in \mathcal{L}(W, V)$. Then $ST = I$ if and only if $TS = I$.

$\dim V = \dim W$

$\boxed{\text{Proof}}$ First, suppose $ST = I$. If $v \in V$ & $Tv = 0$, then $v = Iv = (ST)v = S(Tv) = S(0) = 0$. So, T is injective (since $\operatorname{null} T = \{0\}$). Since $\dim V = \dim W$, T is invertible. So, $T^{-1} = IT^{-1} = (ST)T^{-1} = S(TT^{-1}) = SI = S$ & $S = T^{-1}$. Thus, $TS = TT^{-1} = I$. Now, for the other direction, just reverse the roles of S & T in the previous argument.

Isomorphic Vector Spaces

3.69 definition: *isomorphism, isomorphic*

- An *isomorphism* is an invertible linear map.
- Two vector spaces are called *isomorphic* if there is an isomorphism from one vector space onto the other one.

3.70 *dimension shows whether vector spaces are isomorphic*

Two finite-dimensional vector spaces over **F** are isomorphic if and only if they have the same dimension.

$\boxed{\text{proof}}$ ($\Rightarrow$) V & W are isomorphic. Then, $\exists\, T \in \mathcal{L}(V,W)$ that is invertible and, hence, both injective & surjective. So, null $T = \{0\}$ & range $T = W$. The FToLM then becomes

$$\dim V = \dim \operatorname{null} T + \dim \operatorname{range} T = 0 + \dim W = \dim W.$$

($\Leftarrow$) $\dim V = \dim W < \infty$. Let $v_1, \dots, v_n$ be a basis for V & let $w_1, \dots, w_n$ be a basis for W (i.e. $n = \dim V = \dim W$).

Define $T \in \mathcal{L}(V,W)$ by $Tv = T(c_1v_1 + \dots + c_nv_n) = c_1w_1 + \dots + c_nw_n$ (where $c_1, \dots, c_n \in \mathbb{F}$ are the unique coefficients in the linear combo of $v_1, \dots, v_n$ s.t. $v = c_1v_1 + \dots + c_nv_n$). T is surjective because $w_1, \dots, w_n$ spans W. Furthermore, $Tv = 0 \iff c_1w_1 + \dots + c_nw_n = 0$ which, by linear independence, implies $c_1 = \dots = c_n = 0$ & $v = 0v_1 + \dots + 0v_n = 0$. So, T is injective. Since T is both injective & surjective, it is an isomorphism (i.e. invertible).