3.44 notation: $A_{j,\cdot}$, $A_{\cdot,k}$

Suppose A is an m-by-n matrix.

- If $1 \leq j \leq m$, then $A_{j,\cdot}$ denotes the 1-by-n matrix consisting of row j of A.
- If $1 \leq k \leq n$, then $A_{\cdot,k}$ denotes the m-by-1 matrix consisting of column k of A.

3.46 *entry of matrix product equals row times column*

Suppose A is an m-by-n matrix and B is an n-by-p matrix. Then

$$(AB)_{j,k} = A_{j,\cdot} B_{\cdot,k}$$

if $1 \leq j \leq m$ and $1 \leq k \leq p$. In other words, the entry in row j, column k, of AB equals (row j of A) times (column k of B).

$\boxed{\text{Proof}}$ By definition, $(AB)_{j,k} = A_{j,1} B_{1,k} + \cdots + A_{j,n} B_{n,k}$

$$= \underbrace{\begin{bmatrix} A_{j,1} & \cdots & A_{j,n} \end{bmatrix}}_{A_{j,\cdot}} \underbrace{\begin{bmatrix} B_{1,k} \\ \vdots \\ B_{n,k} \end{bmatrix}}_{B_{\cdot,k}}$$

$$= A_{j,\cdot} B_{\cdot,k}$$

3.48 *column of matrix product equals matrix times column*

Suppose A is an m-by-n matrix and B is an n-by-p matrix. Then

$$(AB)_{\cdot,k} = AB_{\cdot,k}$$

if $1 \leq k \leq p$. In other words, column k of AB equals A times column k of B.

$\boxed{\text{Proof}}$ By the previous result, $(AB)_{j,k} = A_{j,\cdot} B_{\cdot,k}$.

So, $(AB)_{\cdot,k} = A_{\cdot,\cdot} B_{\cdot,k} = AB_{\cdot,k}$.

3.50 *linear combination of columns*

Suppose A is an m-by-n matrix and $b = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$ is an n-by-1 matrix. Then

$$Ab = b_1 A_{\cdot,1} + \cdots + b_n A_{\cdot,n}.$$

In other words, Ab is a linear combination of the columns of A, with the scalars that multiply the columns coming from b.

$\boxed{\text{Proof}}$ By the previous result, $(AB)_{\cdot,k} = A B_{\cdot,k}$.

But, if $B = b$, we have $Ab = (Ab)_{\cdot,1} = A b_{\cdot,1} =$.

This says $\begin{bmatrix} A_{\cdot,1} & \cdots & A_{\cdot,n} \end{bmatrix} \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} = b_1 A_{\cdot,1} + \cdots + b_n A_{\cdot,n}$.

3.51 *matrix multiplication as linear combinations of columns*

Suppose C is an m-by-c matrix and R is a c-by-n matrix.

(a) If $k \in \{1, \ldots, n\}$, then column k of CR is a linear combination of the columns of C, with the coefficients of this linear combination coming from column k of R.

(b) If $j \in \{1, \ldots, m\}$, then row j of CR is a linear combination of the rows of R, with the coefficients of this linear combination coming from row j of C.

Same logic

$\boxed{\text{Proof}}$ S $k \in \{1, \ldots, n\}$. Then, column k of CR i.e. $(CR)_{\cdot,k}$, is equal to a linear combo of the columns of C w/ coefficients coming from $R_{\cdot,k}$ (i.e. the k^{th} column of R) by the previous result: $(CR)_{\cdot,k} = C(R_{\cdot,k}) = \begin{bmatrix} C_{\cdot,1} & \cdots & C_{\cdot,n} \end{bmatrix} \begin{bmatrix} R_{1,k} \\ \vdots \\ R_{n,k} \end{bmatrix} = R_{1k} C_{\cdot,1} + \cdots + R_{nk} C_{\cdot,n}$

Column–Row Factorization and Rank of a Matrix

3.52 definition: *column rank, row rank*

Suppose A is an m-by-n matrix with entries in $\mathbf{F}$.

- The *column rank* of A is the dimension of the span of the columns of A in $\mathbf{F}^{m,1}$.
- The *row rank* of A is the dimension of the span of the rows of A in $\mathbf{F}^{1,n}$.

i.e. column rank A $=$ $\dim \operatorname{Span}(A_{\cdot,1}, \dots, A_{\cdot,n})$

EX

Suppose

$$A = \begin{pmatrix} 4 & 7 & 1 & 8 \\ 3 & 5 & 2 & 9 \end{pmatrix}.$$

The column rank of A is the dimension of

$$\dim\left(\operatorname{span}\left(\begin{pmatrix} 4 \\ 3 \end{pmatrix}, \begin{pmatrix} 7 \\ 5 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 8 \\ 9 \end{pmatrix}\right)\right) = 2$$

The row rank of A is the dimension of

$$\dim\left(\operatorname{span}\left(\begin{pmatrix} 4 & 7 & 1 & 8 \end{pmatrix}, \begin{pmatrix} 3 & 5 & 2 & 9 \end{pmatrix}\right)\right) = 2$$

These seem to be the same!

Is this always so?

3.54 definition: *transpose,* A^{t}

The *transpose* of a matrix A, denoted by A^{t}, is the matrix obtained from A by interchanging rows and columns. Specifically, if A is an m-by-n matrix, then A^{t} is the n-by-m matrix whose entries are given by the equation

$$(A^{\mathrm{t}})_{k,j} = A_{j,k}.$$

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}^t = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$$

3.56 *column–row factorization*

Suppose A is an m-by-n matrix with entries in $\mathbf{F}$ and column rank $c \geq 1$. Then there exist an m-by-c matrix C and a c-by-n matrix R, both with entries in $\mathbf{F}$, such that $A = CR$.

$\boxed{\text{Proof}}$ The list $A_{\cdot,1}, \dots, A_{\cdot,n}$ of columns of A can be reduced to a basis of $\operatorname{span}(A_{\cdot,1}, \dots, A_{\cdot,n})$ (i.e. a spanning list in a finite-dim. V.S. can be reduced to a basis). This basis has length $\dim(\operatorname{span}(A_{\cdot,1}, \dots, A_{\cdot,n})) = c$. The c columns that form this basis can be used to make a matrix C which has these basis elements as its columns. This matrix is m-by-c. Since any column in A can be written as a linear combo of the columns of C, let R be the matrix whose k^{th} column is the coefficients such that $A_{\cdot,k} = R_{1,k} C_{\cdot,1} + \dots + R_{c,k} C_{\cdot,c}$. Then, $A = CR$.

3.57 *column rank equals row rank*

Suppose $A \in \mathbf{F}^{m,n}$. Then the column rank of A equals the row rank of A.

$\boxed{\text{Proof}}$ Let c denote the column rank of A. Let $A = CR$ be the factorization assured by the previous theorem. Every row of A is a linear combo of R's rows, and the coefficients in this linear combo come from C's rows. Since R has c rows, this says that the row rank of A is less than or equal to c, its column rank.

Now, do the same to A^t and note that:

$\text{column rank } A = \text{row rank } A^t \leq \text{column rank } A^t = \text{row rank } A$.

So, column rank $A \geq$ row rank A & column rank $A \leq$ row rank A. Hence, they are equal!

3.58 definition: *rank*

The *rank* of a matrix $A \in \mathbf{F}^{m,n}$ is the column rank of A. (or, equivalently, the row rank).