3B Null Spaces and Ranges

Null Space and Injectivity

3.11 definition: *null space*, null T

For $T \in \mathcal{L}(V, W)$, the *null space* of T, denoted by null T, is the subset of V consisting of those vectors that T maps to 0:

$$\operatorname{null} T = \{v \in V : Tv = 0\}.$$

3.13 *the null space is a subspace*

Suppose $T \in \mathcal{L}(V, W)$. Then null T is a subspace of V.

$\boxed{\text{Proof}}$ Since T is linear, $T(0)=0$ (we proved this last time). So, $0 \in \operatorname{null} T$.
Now, if $u, v \in \operatorname{null} T$, then note that, by linearity,
$T(u+v) = Tu + Tv = 0+0 = 0$. So, $u+v \in \operatorname{null} T$ as well.
Finally, if $v \in \operatorname{null} T$ & $\lambda \in \mathbb{F}$, then by linearity, we have
$T(\lambda v) = \lambda Tv = \lambda 0 = 0$. So, null T is a subspace.

3.14 definition: *injective*

A function $T : V \to W$ is called *injective* if $Tu = Tv$ implies $u = v$.

3.15 *injectivity* $\iff$ *null space equals* $\{0\}$

Let $T \in \mathcal{L}(V, W)$. Then T is injective if and only if null $T = \{0\}$.

$\boxed{\text{Proof}}$ First, $\$$ $T \in \mathcal{L}(V,W)$ is injective. Then, since
$T0=0$, null $T = \{0\}$ since anything else would imply $Tv = 0 = T0$
for some $v \neq 0$, contradicting injectivity.
Now, $\$$ $T \in \mathcal{L}(V,W)$ & null $T = \{0\}$. $\$$ $u, v \in V$ & $Tu = Tv$.
Then, by linearity, we have $0 = Tu - Tv = T(u-v)$. So, $u - v \in \operatorname{null} T$.
But, since null $T = \{0\}$, this says $u - v = 0$, i.e. $u = v$. So,
T is injective.

Range and Surjectivity

3.16 definition: *range*

For $T \in \mathcal{L}(V, W)$, the *range* of T is the subset of W consisting of those vectors that are equal to Tv for some $v \in V$:

$$\operatorname{range} T = \{Tv : v \in V\}.$$

3.18 *the range is a subspace*

If $T \in \mathcal{L}(V, W)$, then $\operatorname{range} T$ is a subspace of W.

$\boxed{\text{Proof}}$ Let $T \in \mathcal{L}(V,W)$. First, note that $0 = T0$, so $0 \in \operatorname{range} T$. Now, ζ $w_1, w_2 \in \operatorname{range} T$. Then, $\exists v_1, v_2 \in V$ st. $Tv_1 = w_1$ \$ $Tv_2 = w_2$. Then, by linearity, $T(v_1 + v_2) = Tv_1 + Tv_2 = w_1 + w_2$. So, $w_1 + w_2 \in \operatorname{range} T$ as well. Finally, ζ $w \in \operatorname{range} T$ \$ $\lambda \in \mathbb{F}$. Then, $\exists v \in V$ st. $Tv = w$ and, by linearity, we have $T(\lambda v) = \lambda Tv = \lambda w$. So, $\lambda w \in \operatorname{range} T$ as well.

3.19 definition: *surjective*

A function $T : V \to W$ is called *surjective* if its range equals W.

3.20 example: *surjectivity depends on the target space*

The differentiation map $D \in \mathcal{L}(\mathcal{P}_5(\mathbf{R}))$ defined by $Dp = p'$ is not surjective, because the polynomial x^5 is not in the range of D. However, the differentiation map $S \in \mathcal{L}(\mathcal{P}_5(\mathbf{R}), \mathcal{P}_4(\mathbf{R}))$ defined by $Sp = p'$ is surjective, because its range equals $\mathcal{P}_4(\mathbf{R})$, which is the vector space into which S maps.

Fundamental Theorem of Linear Maps

3.21 *fundamental theorem of linear maps*

Suppose V is finite-dimensional and $T \in \mathcal{L}(V, W)$. Then range T is finite-dimensional and

$$\dim V = \dim \operatorname{null} T + \dim \operatorname{range} T.$$

$\boxed{\text{Proof}}$ Since null $T \leq V$, it is finite dimensional since V is. Let $u_1, \ldots, u_m$ be a basis of null T. Since this is a lin. indep. list in V, it can be extended to a basis of V. Let $u_1, \ldots, u_m, v_1, \ldots, v_n$ be this extension. Then, $\dim V = n + m = n + \dim \operatorname{null} T$. We will show that $\dim \operatorname{range} T = n$ by showing that $Tv_1, \ldots, Tv_n$ is a basis for range T. Let $v \in V$. Since $u_1, \ldots, u_m, v_1, \ldots, v_n$ is a basis of V, $\exists\, a_1, \ldots, a_m, b_1, \ldots, b_n \in \mathbb{F}$ s.t. $v = a_1 u_1 + \cdots + a_m u_m + b_1 v_1 + \cdots + b_n v_n$.

Then, by linearity, we have $Tv = a_1 Tu_1 + \cdots + a_m Tu_m + b_1 Tv_1 + \cdots + b_n Tv_n$.

Since $u_1, \ldots, u_m \in \operatorname{null} T$, $Tu_i = 0$ for $i = 1, \ldots, m$. So, $\underbrace{Tv}_{\in \operatorname{range} T} = b_1 Tv_1 + \cdots + b_n Tv_n$.

Hence, $Tv_1, \ldots, Tv_n$ spans range T. Now, we show $Tv_1, \ldots, Tv_n$ is lin. indep.: Let $c_1, \ldots, c_n \in \mathbb{F}$ be s.t. $c_1 Tv_1 + \cdots + c_n Tv_n = 0$.

Then, by linearity, this says $T(c_1 v_1 + \cdots + c_n v_n) = 0$. So, $c_1 v_1 + \cdots + c_n v_n \in \operatorname{null} T$, and $\exists\, d_1, \ldots, d_m \in \mathbb{F}$ s.t.

$c_1 v_1 + \cdots + c_n v_n = d_1 u_1 + \cdots + d_m u_m$ (since $u_1, \ldots, u_m$ is a basis for null T).

But all the c's & d's must then be 0 since $u_1, \ldots, u_m, v_1, \ldots, v_n$ is a basis for V & is lin. indep. So, $Tv_1, \ldots, Tv_n$ is a lin. indep. list of vectors that spans range T, i.e., it is a basis of range T.

3.22 *linear map to a lower-dimensional space is not injective*

Suppose V and W are finite-dimensional vector spaces such that $\dim V > \dim W$. Then no linear map from V to W is injective.

$\boxed{\text{Proof}}$ Let $T \in \mathcal{L}(V, W)$. Then,

$$\dim \operatorname{null} T = \dim V - \dim \operatorname{range} T$$

(since $\dim V = \dim \operatorname{null} T + \dim \operatorname{range} T$ by the FToLM.)

But, then, $\dim \operatorname{null} T = \dim V - \dim \operatorname{range} T$
$$\geq \dim V - \dim W$$
$$> 0$$

So, $\dim \operatorname{null} T \neq 0$ & therefore $\operatorname{null} T \neq \{0\}$.
So, T is not injective by a previous result.

3.24 *linear map to a higher-dimensional space is not surjective*

Suppose V and W are finite-dimensional vector spaces such that $\dim V < \dim W$. Then no linear map from V to W is surjective.

$\boxed{\text{Proof}}$ Let $T \in \mathcal{L}(V, W)$. Then, by the FToLM, we have

$$\dim \operatorname{range} T = \dim V - \dim \operatorname{null} T$$
$$\leq \dim V$$
$$< \dim W$$

So, $\operatorname{range} T \neq W$ & T cannot be surjective.

Fix n, m positive integers & let $A_{jk} \in \mathbb{F}$ for $j = 1, \ldots, m$ & $k = 1, \ldots, n$. Consider the system

$$\left.\begin{array}{c} A_{11}x_1 + \cdots + A_{1n}x_n = 0 \\ \vdots \\ A_{m1}x_1 + \cdots + A_{mn}x_n = 0 \end{array}\right] \text{rhs}$$

Clearly, $x_1 = \cdots = x_n = 0$ is a solution. Does any other solution exist?

Define $T : \mathbb{F}^n \to \mathbb{F}^m$ by
$$T(x_1, \ldots, x_n) = (A_{11}x_1 + \cdots + A_{1n}x_n, \ldots, A_{m1}x_1 + \cdots + A_{mn}x_n)$$

3.26 *homogeneous system of linear equations*

A homogeneous system of linear equations with more variables than equations has nonzero solutions.

$\boxed{\text{Proof}}$ The map T above is linear, i.e. $T \in \mathcal{L}(\mathbb{F}^n, \mathbb{F}^m)$ (exercise: check this!). So, if $n > m$, we have $n = \dim \mathbb{F}^n > \dim \mathbb{F}^m = m$ & T cannot be injective, i.e. $\operatorname{null} T \neq \{0\}$. So, nontrivial solutions exist.

3.28 *inhomogeneous system of linear equations*

An inhomogeneous system of linear equations with more equations than variables has no solution for some choice of the constant terms.

$\boxed{\text{Proof}}$ The inhomogeneous systems w/ solutions are those ones whose rhs lies in range T. But, if $m > n$, this cannot be possible for every $w \in \mathbb{F}^m$ since T cannot be surjective.