

3A Vector Space of Linear Maps

vector spaces over the same field

3.1 definition: linear map

A linear map from V to W is a function $T: V \rightarrow W$ with the following properties.

additivity

$$T(u + v) = Tu + Tv \text{ for all } u, v \in V.$$

homogeneity

$$T(\lambda v) = \lambda(Tv) \text{ for all } \lambda \in \mathbb{F} \text{ and all } v \in V.$$

3.2 notation: $\mathcal{L}(V, W)$, $\mathcal{L}(V)$

- The set of linear maps from V to W is denoted by $\mathcal{L}(V, W)$.
- The set of linear maps from V to V is denoted by $\mathcal{L}(V)$. In other words, $\mathcal{L}(V) = \mathcal{L}(V, V)$.

EX

- Let $Tv = 0$ for all $v \in V$, i.e. T is the zero map, which is usually also denoted by 0 .

$$\text{Then, } T(v+w) = 0 = 0 + 0 = Tv + Tw.$$

$$\text{Similarly, } T(\lambda v) = 0 = \lambda 0 = \lambda(Tv).$$

So, the zero map is a linear map.

- Let $I: V \rightarrow V$ be defined by $Iv = v$ for all $v \in V$.

$$\text{Then, } I(v+w) = v+w = Iv + Iw \quad \& \quad I(\lambda v) = \lambda v = \lambda(Iv).$$

So, this map (which is called the identity map) is linear too.

- Define $D \in \mathcal{L}(\mathcal{P}(\mathbb{R}))$ by $Dp = p'$. Then, since the derivative is linear, so is D (i.e. $D(p+q) = p' + q'$ & $D(\lambda p) = \lambda p'$).

- Integration also is linear: Define, for example, $T \in \mathcal{L}(\mathcal{P}(\mathbb{R}), \mathbb{R})$ by $Tp = \int_0^1 p(x) dx$. Then, $T(p+q) = \int_0^1 (p(x)+q(x)) dx = \int_0^1 p(x) dx + \int_0^1 q(x) dx = Tp + Tq$ & $T(\lambda p) = \int_0^1 \lambda p(x) dx = \lambda \int_0^1 p(x) dx = \lambda Tp$.

3.4 linear map lemma

Suppose $v_1, \dots, v_n$ is a basis of V and $w_1, \dots, w_n \in W$. Then there exists a unique linear map $T: V \rightarrow W$ such that

!

$$Tv_k = w_k \leftarrow$$

for each $k = 1, \dots, n$.

Proof Let V be a finite-dim. V.S. & $v_1, \dots, v_n$ is a basis for V . Let W be a vector space & let $w_1, \dots, w_n \in W$. Define $T: V \rightarrow W$

$$\text{by } Tv = T(a_1v_1 + \dots + a_nv_n) = a_1w_1 + \dots + a_nw_n$$

(which is possible since $v_1, \dots, v_n$ is a basis & therefore every $v \in V$ can be written uniquely as $v = a_1v_1 + \dots + a_nv_n$ for some $a_1, \dots, a_n \in \mathbb{F}$).

Taking $a_k = 1$ & $a_i = 0$ for $i \neq k$, we get $Tv_k = w_k$.

To see $T \in \mathcal{L}(V, W)$, note that

$$\begin{aligned} T(v+w) &= T(\underbrace{(a_1v_1 + \dots + a_nv_n)}_v + \underbrace{(b_1v_1 + \dots + b_nv_n)}_w) \\ &= T((a_1+b_1)v_1 + \dots + (a_n+b_n)v_n) \\ &= (a_1+b_1)w_1 + \dots + (a_n+b_n)w_n \\ &= (a_1w_1 + \dots + a_nw_n) + (b_1w_1 + \dots + b_nw_n) \\ &= T(a_1v_1 + \dots + a_nv_n) + T(b_1v_1 + \dots + b_nv_n) \\ &= Tv + Tw \end{aligned}$$

$$\begin{aligned} T(\lambda v) &= T(\lambda(a_1v_1 + \dots + a_nv_n)) \\ &= T((\lambda a_1)v_1 + \dots + (\lambda a_n)v_n) \\ &= (\lambda a_1)w_1 + \dots + (\lambda a_n)w_n \\ &= \lambda(a_1w_1 + \dots + a_nw_n) \\ &= \lambda T(a_1v_1 + \dots + a_nv_n) \\ &= \lambda Tv \end{aligned}$$

So, T is linear. For uniqueness, & $T' \in \mathcal{L}(V, W)$ satisfies $T'v_k = w_k$.

Then, $T'(v) = T'(a_1v_1 + \dots + a_nv_n) = a_1(T'v_1) + \dots + a_n(T'v_n) = a_1w_1 + \dots + a_nw_n = T(v)$.

So, $T' = T$ since they agree at every $v \in V$.

Algebraic Operations on $\mathcal{L}(V, W)$

3.5 definition: addition and scalar multiplication on $\mathcal{L}(V, W)$

Suppose $S, T \in \mathcal{L}(V, W)$ and $\lambda \in \mathbb{F}$. The sum $S + T$ and the product λT are the linear maps from V to W defined by

$$(S + T)(v) = Sv + Tv \quad \text{and} \quad (\lambda T)(v) = \lambda(Tv)$$

for all $v \in V$.

linearity:

$$\begin{aligned} (S+T)(v+w) &= S(v+w) + T(v+w) \\ &= Sv + Sw + Tv + Tw \\ &= Sv + Tv + Sw + Tw \\ &= (S+T)(v) + (S+T)(w) \end{aligned}$$

$$\begin{aligned} (S+T)(\lambda v) &= S(\lambda v) + T(\lambda v) \\ &= \lambda(Sv) + \lambda(Tv) \\ &= \lambda(Sv + Tv) \\ &= \lambda(S+T)(v) \end{aligned}$$

3.6 $\mathcal{L}(V, W)$ is a vector space

With the operations of addition and scalar multiplication as defined above, $\mathcal{L}(V, W)$ is a vector space.

Proof Routine — leave as an exercise!

3.7 definition: product of linear maps

If $T \in \mathcal{L}(U, V)$ and $S \in \mathcal{L}(V, W)$, then the product $ST \in \mathcal{L}(U, W)$ is defined by

$$(ST)(u) = S(Tu)$$

for all $u \in U$.

$$u \xrightarrow{T} V \xrightarrow{S} W$$

$\searrow$
 ST

3.8 algebraic properties of products of linear maps

associativity

$(T_1 T_2) T_3 = T_1 (T_2 T_3)$ whenever T_1, T_2 , and T_3 are linear maps such that the products make sense (meaning T_3 maps into the domain of T_2 , and T_2 maps into the domain of T_1).

identity

$TI = IT = T$ whenever $T \in \mathcal{L}(V, W)$; here the first I is the identity operator on V , and the second I is the identity operator on W .

distributive properties

$(S_1 + S_2)T = S_1 T + S_2 T$ and $S(T_1 + T_2) = ST_1 + ST_2$ whenever $T, T_1, T_2 \in \mathcal{L}(U, V)$ and $S, S_1, S_2 \in \mathcal{L}(V, W)$.

$$u \xrightarrow{T_1} V \xrightarrow{T_2} W \xrightarrow{T_3} Y$$

Proof Exercise — routine

3.10 linear maps take 0 to 0

Suppose T is a linear map from V to W . Then $T(0) = 0$.

Proof $\S T \in \mathcal{L}(V, W)$. Then, we have

$$T(0) = T(0+0) \quad (\text{since } 0 \text{ is the additive id. of } V)$$

$$T(0) = T(0) + T(0) \quad (\text{by linearity of } T)$$

$$-T(0) + T(0) = -T(0) + (T(0) + T(0)) \quad (\text{add } -T(0) \text{ by existence of addition inverses in } W)$$

$$0 = 0 + T(0) \quad (\text{def'n of addition inverses})$$

$$0 = T(0) \quad (\text{def'n of addition identity})$$