

2A Span and Linear Independence

2.2 definition: linear combination

A linear combination of a list $v_1, \dots, v_m$ of vectors in V is a vector of the form

$$a_1 v_1 + \dots + a_m v_m,$$

where $a_1, \dots, a_m \in \mathbb{F}$.

2.4 definition: span

The set of all linear combinations of a list of vectors $v_1, \dots, v_m$ in V is called the span of $v_1, \dots, v_m$, denoted by $\text{span}(v_1, \dots, v_m)$. In other words,

$$\text{span}(v_1, \dots, v_m) = \{a_1 v_1 + \dots + a_m v_m : a_1, \dots, a_m \in \mathbb{F}\}.$$

The span of the empty list $()$ is defined to be $\{0\}$. $\leftarrow$ convention

2.6 span is the smallest containing subspace

The span of a list of vectors in V is the smallest subspace of V containing all vectors in the list.

Proof Notice: If $v_1, \dots, v_m \in V$ is a list of vectors, then $0v_1 + \dots + 0v_m = 0$, so $0 \in \text{span}(v_1, \dots, v_m)$.

Moreover, if $v \in \text{span}(v_1, \dots, v_m)$, then $v = a_1 v_1 + \dots + a_m v_m$ for some $a_1, \dots, a_m \in \mathbb{F}$. For any $b \in \mathbb{F}$, we have $bv = b(a_1 v_1 + \dots + a_m v_m) = (ba_1)v_1 + \dots + (ba_m)v_m \in \text{span}(v_1, \dots, v_m)$.

Similarly, if $v' = a'_1 v_1 + \dots + a'_m v_m \in \text{span}(v_1, \dots, v_m)$, then

$$\begin{aligned} v + v' &= (a_1 v_1 + \dots + a_m v_m) + (a'_1 v_1 + \dots + a'_m v_m) \\ &= (a_1 v_1 + a'_1 v_1) + \dots + (a_m v_m + a'_m v_m) \\ &= (a_1 + a'_1)v_1 + \dots + (a_m + a'_m)v_m \in \text{span}(v_1, \dots, v_m) \end{aligned}$$

So, by a theorem we proved last time, $\text{span}(v_1, \dots, v_m)$ is a subspace of V . Moreover, notice that

$$v_i = 0v_1 + \dots + 0v_{i-1} + (1)v_i + 0v_{i+1} + \dots + 0v_m, \text{ so } v_i \in \text{span}(v_1, \dots, v_m)$$

for $i=1, \dots, m$.

Now, for $W \subseteq V$ and $v_1, \dots, v_m \in W$. Then, since linear combos of $v_1, \dots, v_m$ are certainly also contained in W , $\text{span}(v_1, \dots, v_m) \subseteq W$ as well.

$\text{span}(v_1, \dots, v_m)$ is a Subspace containing $v_1, \dots, v_m$

It is the smallest one.

2.7 definition: *spans*

If $\text{span}(v_1, \dots, v_m)$ equals V , we say that the list $v_1, \dots, v_m$ spans V .

2.9 definition: *finite-dimensional vector space*

A vector space is called finite-dimensional if some list of vectors in it spans the space.

Recall that by definition every list has finite length.

2.13 definition: *infinite-dimensional vector space*

A vector space is called infinite-dimensional if it is not finite-dimensional.

2.10 definition: *polynomial, $\mathcal{P}(\mathbb{F})$*

- A function $p: \mathbb{F} \rightarrow \mathbb{F}$ is called a polynomial with coefficients in $\mathbb{F}$ if there exist $a_0, \dots, a_m \in \mathbb{F}$ such that

$$p(z) = a_0 + a_1 z + a_2 z^2 + \dots + a_m z^m$$

for all $z \in \mathbb{F}$.

- $\mathcal{P}(\mathbb{F})$ is the set of all polynomials with coefficients in $\mathbb{F}$.

2.12 notation: $\mathcal{P}_m(\mathbb{F})$

For m a nonnegative integer, $\mathcal{P}_m(\mathbb{F})$ denotes the set of all polynomials with coefficients in $\mathbb{F}$ and degree at most m .

Ex $\mathcal{P}_m(\mathbb{F}) \subseteq \mathcal{P}(\mathbb{F})$

i.e. add together two polys of degree ^{at most} m ,
get back a poly. of degree at most m .
Similarly, scale a poly. of degree at most m ,
get back a poly. of the same degree or, if
the scalar is 0, a lower degree poly.

Notice: $\mathcal{P}_m(\mathbb{F}) = \text{Span}(1, z, z^2, \dots, z^m)$

So, $\mathcal{P}_m(\mathbb{F})$ is finite dimensional!

However, $\mathcal{P}(\mathbb{F})$ is not!

2.15 definition: linearly independent

- A list $v_1, \dots, v_m$ of vectors in V is called linearly independent if the only choice of $a_1, \dots, a_m \in \mathbb{F}$ that makes

$$a_1 v_1 + \dots + a_m v_m = 0$$

is $a_1 = \dots = a_m = 0$.

- The empty list $()$ is also declared to be linearly independent.

2.17 definition: linearly dependent

- A list of vectors in V is called linearly dependent if it is not linearly independent.
- In other words, a list $v_1, \dots, v_m$ of vectors in V is linearly dependent if there exist $a_1, \dots, a_m \in \mathbb{F}$, not all 0, such that $a_1 v_1 + \dots + a_m v_m = 0$.

2.19 linear dependence lemma

Suppose $v_1, \dots, v_m$ is a linearly dependent list in V . Then there exists $k \in \{1, 2, \dots, m\}$ such that

$$v_k \in \text{span}(v_1, \dots, v_{k-1}).$$

Furthermore, if k satisfies the condition above and the k^{th} term is removed from $v_1, \dots, v_m$, then the span of the remaining list equals $\text{span}(v_1, \dots, v_m)$.

Proof Since $v_1, \dots, v_m$ is lin. dep., there are scalars $a_1, \dots, a_m \in \mathbb{F}$, not all zero, such that $a_1 v_1 + \dots + a_m v_m = 0$. Let k be the largest # st. $a_k \neq 0$. Then, $0 = a_1 v_1 + \dots + a_k v_k + a_{k+1} v_{k+1} + \dots + a_m v_m$
 $0 = a_1 v_1 + \dots + a_k v_k + 0 + \dots + 0$

So, $-a_k v_k = a_1 v_1 + \dots + a_{k-1} v_{k-1}$ and, since $a_k \neq 0$, we can multiply by $(-a_k)^{-1}$ to get

$$v_k = (-a_k)^{-1} a_1 v_1 + \dots + (-a_k)^{-1} a_{k-1} v_{k-1}$$

Hence, $v_k \in \text{span}(v_1, \dots, v_{k-1})$. To see the rest, notice that this lets us rewrite any scaling of v_k as a linear combo of $v_1, \dots, v_{k-1}$.

In a finite-dimensional vector space, the length of every linearly independent list of vectors is less than or equal to the length of every spanning list of vectors.

Proof Let $\text{span}(w_1, \dots, w_n) = V$, i.e. $\{w_1, \dots, w_n\}$ is a spanning list. Let $\underline{u_1, \dots, u_m}$ be a linearly indep. list. Since $w_1, \dots, w_n$ spans V , so does $\underline{u_1, w_1, \dots, w_n}$. However, since $u_i \in V$, $u_1, w_1, \dots, w_n$ is linearly dependent. So, by the lin. dep. lemma, $\exists K$ st. the list $\underline{u_1, w_1, \dots, w_{K-1}, w_{K+1}, \dots, w_n}$ spans V .
length n

This is step 1. Here is step K for $K=2, \dots, m$:

Take your list from the previous step & notice that u_K is in its span. So, adding u_K in the K^{th} position in the list yields a lin. dep. list & we can use the lin. dep. lemma to remove an element. Since the start of this list is all u 's & $u_1, \dots, u_m$ is lin. indep., we get that the removed element must be a w_i .

Thus, at the end of this process, we get a list that looks like $\underline{u_1, \dots, u_m}, w_{t_1}, w_{t_2}, \dots, w_{t_{n-m}}$

(where there are no w 's in the case where $n=m$) that spans V .

So, the length of such a list must be at least as long as the length of the lin. indep. list $u_1, \dots, u_m$ (i.e. m).

Every subspace of a finite-dimensional vector space is finite-dimensional.

Proof Let V be a fin. dim. V.S.. Then,

Let $U \leq V$.

Step 1: If $U = \{0\}$, we are done since $U = \text{span}(\)$.
If not, let $u_1 \in U$ be nonzero.

Step j : Take $\text{span}(u_1, \dots, u_{j-1})$. If this is equal to U ,
for $j \in \{2, \dots, k\}$ we are done. If not, take some $u_j \notin \text{span}(u_1, \dots, u_{j-1})$
& repeat now with $\text{span}(u_1, \dots, u_j)$ as the
next step.

Now, since $u_j \notin \text{span}(u_1, \dots, u_{j-1})$, the list
 $u_1, \dots, u_j$ is lin. indep. By the previous
result, the length of every lin. indep. list in U
is less than or equal to the length of any list
that spans U . Since U is finite dim., there
is a finite length list that spans U , so there is an
upper bound on how long a lin. indep. list in U can be.

This applies to the u_i 's — the process has to
eventually stop at some k that is less than or
equal to the length of any choice of a finite spanning list for U .
Hence, we get $U = \text{span}(u_1, \dots, u_k)$ & U is
finite dim.