

IC Subspaces

$U \subseteq V \leftarrow$ subset
 $U \leq V \leftarrow$ subspace.
 We write $U \leq V$ to denote that U is a subspace.

1.33 definition: subspace

A subset U of V is called a subspace of V if U is also a vector space with the same additive identity, addition, and scalar multiplication as on V .

1.34 conditions for a subspace

A subset U of V is a subspace of V if and only if U satisfies the following three conditions.

① additive identity

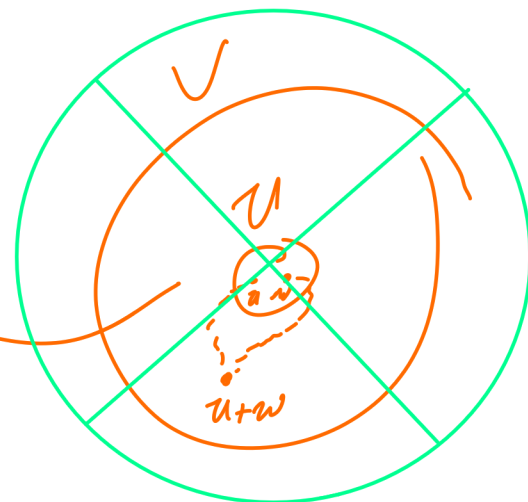
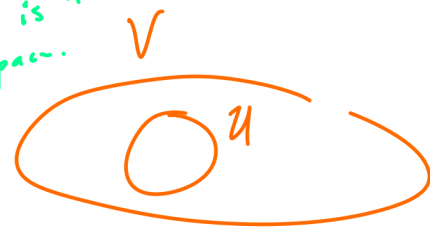
$$0 \in U.$$

② closed under addition

$$u, w \in U \text{ implies } u + w \in U.$$

③ closed under scalar multiplication

$$a \in \mathbb{F} \text{ and } u \in U \text{ implies } au \in U.$$



Proof (Sketch)

If U is a subspace, then this holds by definition.

In the reverse, notice that any of these being violated would prevent U from being a subspace. Moreover, they are precisely the 3 conditions defining a subspace.

1.36 definition: sum of subspaces

Suppose $V_1, \dots, V_m$ are subspaces of V . The sum of $V_1, \dots, V_m$, denoted by $V_1 + \dots + V_m$, is the set of all possible sums of elements of $V_1, \dots, V_m$. More precisely,

$$V_1 + \dots + V_m = \{v_1 + \dots + v_m : v_1 \in V_1, \dots, v_m \in V_m\}.$$

Ex $U = \{ (x, x, y, y) \in \mathbb{F}^4 : x, y \in \mathbb{F} \}$

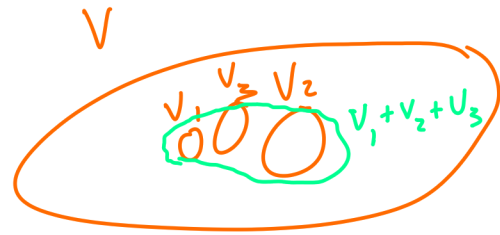
$$(x_1, x_1, y_1, y_1) + (x_2, x_2, y_2, y_2) = (\underbrace{x_1 + x_2}_{\text{Same}}, \underbrace{x_1 + x_2}_{\text{Same}}, \underbrace{y_1 + y_2}_{\text{Same}}, \underbrace{y_1 + y_2}_{\text{Same}})$$

$$W = \{ (x, x, x, y) \in \mathbb{F}^4 : x, y \in \mathbb{F} \}$$

$$U + W = \{ (x, x, y, z) \in \mathbb{F}^4 : x, y, z \in \mathbb{F} \}$$

1.40 sum of subspaces is the smallest containing subspace

Suppose $V_1, \dots, V_m$ are subspaces of V . Then $V_1 + \dots + V_m$ is the smallest subspace of V containing $V_1, \dots, V_m$.



Proof

Clearly, $V_i \subseteq V_1 + \dots + V_m$, since

$v_i = 0 + \dots + 0 + v_i + 0 + \dots + 0 \in V_1 + \dots + V_m$ for any $v_i \in V_i$.

Conversely, if $W \subseteq V$ such that $V_i \subseteq W$ for $i=1, \dots, m$, then $V_1 + \dots + V_m \subseteq W$ as well, as subspaces must contain all finite sums of their elements.

1.41 definition: direct sum, $\oplus$

Suppose $V_1, \dots, V_m$ are subspaces of V .

- The sum $V_1 + \dots + V_m$ is called a direct sum if each element of $V_1 + \dots + V_m$ can be written in only one way as a sum $v_1 + \dots + v_m$, where each $v_k \in V_k$.
- If $V_1 + \dots + V_m$ is a direct sum, then $V_1 \oplus \dots \oplus V_m$ denotes $V_1 + \dots + V_m$, with the $\oplus$ notation serving as an indication that this is a direct sum.

[EX]

Let $U \neq \{0\} \subseteq V$. Then,

$U+U$ is not direct!

If $w \in U+U$, then notice that $w \in U$, so $w = 0 + w = w + 0$ i.e. $w = u_1 + u_2 = u'_1 + u'_2$ for $u'_i \neq u_i$.

1.45 condition for a direct sum

Suppose $V_1, \dots, V_m$ are subspaces of V . Then $V_1 + \dots + V_m$ is a direct sum if and only if the only way to write 0 as a sum $v_1 + \dots + v_m$, where each $v_k \in V_k$, is by taking each v_k equal to 0.

Proof

If $V_1 + \dots + V_m$ is a direct sum, then

since $0 = 0 + \dots + 0$, we get our desired result.

Now, suppose the only way to write $0 \in V_1 + \dots + V_m$ is as

$0 = 0 + \dots + 0$, i.e., $0 = v_1 + \dots + v_m$ implies $v_i = 0$ for $i = 1, \dots, m$.

Now, $\$ w = v_1 + \dots + v_m \quad \& \quad w = v'_1 + \dots + v'_m$ for some $w \in V_1 + \dots + V_m$.

$$\begin{aligned} \text{Then, } 0 &= w - w = (v_1 + \dots + v_m) - (v'_1 + \dots + v'_m) \\ &= (v_1 - v'_1) + \dots + (v_m - v'_m) \end{aligned}$$

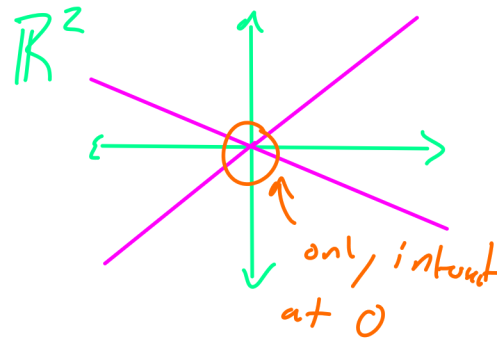
Since $v_i - v'_i \in V_i$ for $i = 1, \dots, m$, we get $v_i - v'_i = 0$

Hence, $v_i = v'_i$ and there is only one way to write w as a sum of elements of V_i for $i = 1, \dots, m$. That is,

$V_1 + \dots + V_m$ is direct.

Suppose U and W are subspaces of V . Then

$$U + W \text{ is a direct sum} \iff U \cap W = \{0\}.$$



Proof

\$ $U+W$ is direct. If $v \in U \cap W$, then $0 = v + (-v)$

Since $v \in U \cap W \Rightarrow v \in U$ & $v \in W$ (and, hence, $-v \in W$).

So, by the previous result, $v = -v = 0$. Hence, $U \cap W = \{0\}$.

Now, \$ $U \cap W = \{0\}$. Then, if $u \in U$ & $w \in W$ are such that $u+w=0$, then $u=-w$. Hence, $u \in W$ as well (since W is a vector space). But, only 0 belongs to both W & U

So $w=0$. Hence, $u+w=0 \Rightarrow u=w=0$ & by the previous result, $U+W$ is direct.

