

7E Singular Value Decomposition

7.64 properties of T^*T

Suppose $T \in \mathcal{L}(V, W)$. Then

- (a) T^*T is a positive operator on V ;
- (b) $\text{null } T^*T = \text{null } T$;
- (c) $\text{range } T^*T = \text{range } T^*$;
- (d) $\dim \text{range } T = \dim \text{range } T^* = \dim \text{range } T^*T$.

7.65 definition: *singular values*

Suppose $T \in \mathcal{L}(V, W)$. The *singular values* of T are the nonnegative square roots of the eigenvalues of T^*T , listed in decreasing order, each included as many times as the dimension of the corresponding eigenspace of T^*T .

7.68 *role of positive singular values*

Suppose that $T \in \mathcal{L}(V, W)$. Then

- (a) T is injective $\iff 0$ is not a singular value of T ;
- (b) the number of positive singular values of T equals $\dim \text{range } T$;
- (c) T is surjective $\iff$ number of positive singular values of T equals $\dim W$.

list of eigenvalues	list of singular values
context: vector spaces	context: inner product spaces
defined only for linear maps from a vector space to itself	defined for linear maps from an inner product space to a possibly different inner product space
can be arbitrary real numbers (if $\mathbf{F} = \mathbf{R}$) or complex numbers (if $\mathbf{F} = \mathbf{C}$)	are nonnegative numbers
can be the empty list if $\mathbf{F} = \mathbf{R}$	length of list equals dimension of domain
includes $0 \iff$ operator is not invertible	includes $0 \iff$ linear map is not injective
no standard order, especially if $\mathbf{F} = \mathbf{C}$	always listed in decreasing order

7.69 *isometries characterized by having all singular values equal 1*

Suppose that $S \in \mathcal{L}(V, W)$. Then

S is an isometry $\iff$ all singular values of S equal 1.

7.70 *singular value decomposition*

Suppose $T \in \mathcal{L}(V, W)$ and the positive singular values of T are $s_1, \dots, s_m$. Then there exist orthonormal lists $e_1, \dots, e_m$ in V and $f_1, \dots, f_m$ in W such that

$$7.71 \quad Tv = s_1 \langle v, e_1 \rangle f_1 + \dots + s_m \langle v, e_m \rangle f_m$$

for every $v \in V$.

spectral theorem	singular value decomposition
describes only self-adjoint operators (when $\mathbf{F} = \mathbf{R}$) or normal operators (when $\mathbf{F} = \mathbf{C}$)	describes arbitrary linear maps from an inner product space to a possibly different inner product space
produces a single orthonormal basis	produces two orthonormal lists, one for domain space and one for range space, that are not necessarily the same even when range space equals domain space
different proofs depending on whether $\mathbf{F} = \mathbf{R}$ or $\mathbf{F} = \mathbf{C}$	same proof works regardless of whether $\mathbf{F} = \mathbf{R}$ or $\mathbf{F} = \mathbf{C}$

7.75 singular value decomposition of adjoint and pseudoinverse

Suppose $T \in \mathcal{L}(V, W)$ and the positive singular values of T are $s_1, \dots, s_m$. Suppose $e_1, \dots, e_m$ and $f_1, \dots, f_m$ are orthonormal lists in V and W such that

$$7.76 \quad Tv = s_1 \langle v, e_1 \rangle f_1 + \dots + s_m \langle v, e_m \rangle f_m$$

for every $v \in V$. Then

$$7.77 \quad T^*w = s_1 \langle w, f_1 \rangle e_1 + \dots + s_m \langle w, f_m \rangle e_m$$

and

$$7.78 \quad T^\dagger w = \frac{\langle w, f_1 \rangle}{s_1} e_1 + \dots + \frac{\langle w, f_m \rangle}{s_m} e_m$$

for every $w \in W$.

7.80 matrix version of SVD

Suppose A is an M -by- n matrix of rank $m \geq 1$. Then there exist an M -by- m matrix B with orthonormal columns, an m -by- m diagonal matrix D with positive numbers on the diagonal, and an n -by- m matrix C with orthonormal columns such that

$$A = BDC^*.$$