

7C Positive Operators

7.34 definition: *positive operator*

An operator $T \in \mathcal{L}(V)$ is called *positive* if T is self-adjoint and

$$\langle Tv, v \rangle \geq 0$$

for all $v \in V$.

7.36 definition: *square root*

An operator R is called a *square root* of an operator T if $R^2 = T$.

7.38 *characterization of positive operators*

Let $T \in \mathcal{L}(V)$. Then the following are equivalent.

- (a) T is a positive operator.
- (b) T is self-adjoint and all eigenvalues of T are nonnegative.
- (c) With respect to some orthonormal basis of V , the matrix of T is a diagonal matrix with only nonnegative numbers on the diagonal.
- (d) T has a positive square root.
- (e) T has a self-adjoint square root.
- (f) $T = R^*R$ for some $R \in \mathcal{L}(V)$.

7.39 *each positive operator has only one positive square root*

Every positive operator on V has a unique positive square root.

7.40 notation: $\sqrt{T}$

For T a positive operator, $\sqrt{T}$ denotes the unique positive square root of T .

7.43 T positive and $\langle Tv, v \rangle = 0 \implies Tv = 0$

Suppose T is a positive operator on V and $v \in V$ is such that $\langle Tv, v \rangle = 0$. Then $Tv = 0$.