7A Self-Adjoint and Normal Operators

Adjoints

7.1 definition: *adjoint*, T^*

Suppose $T \in \mathcal{L}(V, W)$. The *adjoint* of T is the function $T^* \colon W \to V$ such that

$$\langle Tv, w \rangle = \langle v, T^* w \rangle$$

for every $v \in V$ and every $w \in W$.

7.4 *adjoint of a linear map is a linear map*

If $T \in \mathcal{L}(V, W)$, then $T^* \in \mathcal{L}(W, V)$.

7.5 *properties of the adjoint*

Suppose $T \in \mathcal{L}(V, W)$. Then

(a) $(S + T)^* = S^* + T^*$ for all $S \in \mathcal{L}(V, W)$;

(b) $(\lambda T)^* = \overline{\lambda} T^*$ for all $\lambda \in \mathbf{F}$;

(c) $(T^*)^* = T$;

(d) $(ST)^* = T^* S^*$ for all $S \in \mathcal{L}(W, U)$ (here U is a finite-dimensional inner product space over $\mathbf{F}$);

(e) $I^* = I$, where I is the identity operator on V;

(f) if T is invertible, then T^* is invertible and $(T^*)^{-1} = (T^{-1})^*$.

7.6 *null space and range of T^**

Suppose $T \in \mathcal{L}(V, W)$. Then

(a) $\operatorname{null} T^* = (\operatorname{range} T)^\perp$;

(b) $\operatorname{range} T^* = (\operatorname{null} T)^\perp$;

(c) $\operatorname{null} T = (\operatorname{range} T^*)^\perp$;

(d) $\operatorname{range} T = (\operatorname{null} T^*)^\perp$.

7.7 definition: *conjugate transpose, A^**

The *conjugate transpose* of an m-by-n matrix A is the n-by-m matrix A^* obtained by interchanging the rows and columns and then taking the complex conjugate of each entry. In other words, if $j \in \{1, \dots, n\}$ and $k \in \{1, \dots, m\}$, then

$$(A^*)_{j,k} = \overline{A_{k,j}}.$$

7.9 *matrix of T^* equals conjugate transpose of matrix of T*

Let $T \in \mathcal{L}(V, W)$. Suppose $e_1, \dots, e_n$ is an orthonormal basis of V and $f_1, \dots, f_m$ is an orthonormal basis of W. Then $\mathcal{M}(T^*, (f_1, \dots, f_m), (e_1, \dots, e_n))$ is the conjugate transpose of $\mathcal{M}(T, (e_1, \dots, e_n), (f_1, \dots, f_m))$. In other words,

$$\mathcal{M}(T^*) = (\mathcal{M}(T))^*.$$

Self-Adjoint Operators

7.10 definition: *self-adjoint*

An operator $T \in \mathcal{L}(V)$ is called *self-adjoint* if $T = T^*$.

An operator $T \in \mathcal{L}(V)$ is self-adjoint if and only if

$$\langle Tv, w \rangle = \langle v, Tw \rangle$$

for all $v, w \in V$.

7.12 *eigenvalues of self-adjoint operators*

Every eigenvalue of a self-adjoint operator is real.

7.13 *Tv is orthogonal to v for all $v \iff T = 0$ (assuming $\mathbf{F} = \mathbf{C}$)*

Suppose V is a complex inner product space and $T \in \mathcal{L}(V)$. Then

$$\langle Tv, v \rangle = 0 \text{ for every } v \in V \iff T = 0.$$