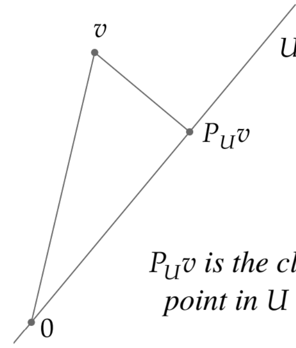
Minimization Problems

6.61 *minimizing distance to a subspace*

Suppose U is a finite-dimensional subspace of V, $v \in V$, and $u \in U$. Then

$$\|v - P_U v\| \le \|v - u\|.$$

Furthermore, the inequality above is an equality if and only if $u = P_U v$.



$P_U v$ is the closest point in U to v.

Pseudoinverse

6.67 *restriction of a linear map to obtain a one-to-one and onto map*

Suppose V is finite-dimensional and $T \in \mathcal{L}(V, W)$. Then $T|_{(\operatorname{null} T)^\perp}$ is a one-to-one map of $(\operatorname{null} T)^\perp$ onto range T.

6.68 definition: *pseudoinverse, $T^\dagger$*

Suppose that V is finite-dimensional and $T \in \mathcal{L}(V, W)$. The *pseudoinverse* $T^\dagger \in \mathcal{L}(W, V)$ of T is the linear map from W to V defined by

$$T^\dagger w = (T|_{(\operatorname{null} T)^\perp})^{-1} P_{\operatorname{range} T}\, w$$

for each $w \in W$.

6.69 *algebraic properties of the pseudoinverse*

Suppose V is finite-dimensional and $T \in \mathcal{L}(V, W)$.

(a) If T is invertible, then $T^\dagger = T^{-1}$.

(b) $TT^\dagger = P_{\operatorname{range} T} =$ the orthogonal projection of W onto range T.

(c) $T^\dagger T = P_{(\operatorname{null} T)^\perp} =$ the orthogonal projection of V onto $(\operatorname{null} T)^\perp$.

6.70 *pseudoinverse provides best approximate solution or best solution*

Suppose V is finite-dimensional, $T \in \mathcal{L}(V, W)$, and $b \in W$.

(a) If $x \in V$, then

$$\|T(T^\dagger b) - b\| \leq \|Tx - b\|,$$

with equality if and only if $x \in T^\dagger b + \operatorname{null} T$.

(b) If $x \in T^\dagger b + \operatorname{null} T$, then

$$\|T^\dagger b\| \leq \|x\|,$$

with equality if and only if $x = T^\dagger b$.