6B Orthonormal Bases

Orthonormal Lists and the Gram–Schmidt Procedure

6.22 definition: *orthonormal*

- A list of vectors is called *orthonormal* if each vector in the list has norm 1 and is orthogonal to all the other vectors in the list.
- In other words, a list $e_1, \ldots, e_m$ of vectors in V is orthonormal if

$$\langle e_j, e_k \rangle = \begin{cases} 1 & \text{if } j = k, \\ 0 & \text{if } j \neq k \end{cases}$$

for all $j, k \in \{1, \ldots, m\}$.

6.24 *norm of an orthonormal linear combination*

Suppose $e_1, \ldots, e_m$ is an orthonormal list of vectors in V. Then

$$\|a_1 e_1 + \cdots + a_m e_m\|^2 = |a_1|^2 + \cdots + |a_m|^2$$

for all $a_1, \ldots, a_m \in \mathbf{F}$.

6.25 *orthonormal lists are linearly independent*

Every orthonormal list of vectors is linearly independent.

6.26 *Bessel's inequality*

Suppose $e_1, \ldots, e_m$ is an orthonormal list of vectors in V. If $v \in V$ then

$$|\langle v, e_1 \rangle|^2 + \cdots + |\langle v, e_m \rangle|^2 \le \|v\|^2.$$

6.27 definition: *orthonormal basis*

An *orthonormal basis* of V is an orthonormal list of vectors in V that is also a basis of V.

6.28 *orthonormal lists of the right length are orthonormal bases*

Suppose V is finite-dimensional. Then every orthonormal list of vectors in V of length $\dim V$ is an orthonormal basis of V.

6.30 *writing a vector as a linear combination of an orthonormal basis*

Suppose $e_1, \ldots, e_n$ is an orthonormal basis of V and $u, v \in V$. Then

(a) $v = \langle v, e_1 \rangle e_1 + \cdots + \langle v, e_n \rangle e_n$,

(b) $\|v\|^2 = |\langle v, e_1 \rangle|^2 + \cdots + |\langle v, e_n \rangle|^2$,

(c) $\langle u, v \rangle = \langle u, e_1 \rangle \overline{\langle v, e_1 \rangle} + \cdots + \langle u, e_n \rangle \overline{\langle v, e_n \rangle}$.

6.32 *Gram–Schmidt procedure*

Suppose $v_1, \ldots, v_m$ is a linearly independent list of vectors in V. Let $f_1 = v_1$. For $k = 2, \ldots, m$, define f_k inductively by

$$f_k = v_k - \frac{\langle v_k, f_1 \rangle}{\|f_1\|^2} f_1 - \cdots - \frac{\langle v_k, f_{k-1} \rangle}{\|f_{k-1}\|^2} f_{k-1}.$$

For each $k = 1, \ldots, m$, let $e_k = \dfrac{f_k}{\|f_k\|}$. Then $e_1, \ldots, e_m$ is an orthonormal list of vectors in V such that

$$\operatorname{span}(v_1, \ldots, v_k) = \operatorname{span}(e_1, \ldots, e_k)$$

for each $k = 1, \ldots, m$.

6.35 *existence of orthonormal basis*

Every finite-dimensional inner product space has an orthonormal basis.

6.36 *every orthonormal list extends to an orthonormal basis*

Suppose V is finite-dimensional. Then every orthonormal list of vectors in V can be extended to an orthonormal basis of V.

6.37 *upper-triangular matrix with respect to some orthonormal basis*

Suppose V is finite-dimensional and $T \in \mathcal{L}(V)$. Then T has an upper-triangular matrix with respect to some orthonormal basis of V if and only if the minimal polynomial of T equals $(z - \lambda_1)\cdots(z - \lambda_m)$ for some $\lambda_1, \ldots, \lambda_m \in \mathbf{F}$.

6.38 *Schur's theorem*

Every operator on a finite-dimensional complex inner product space has an upper-triangular matrix with respect to some orthonormal basis.