

5A Invariant Subspaces

5.1 definition: *operator*

A linear map from a vector space to itself is called an *operator*.

Recall that we defined the notation $\mathcal{L}(V)$ to mean $\mathcal{L}(V, V)$.

5.2 definition: *invariant subspace*

Suppose $T \in \mathcal{L}(V)$. A subspace U of V is called *invariant* under T if $Tu \in U$ for every $u \in U$.

5.5 definition: *eigenvalue*

Suppose $T \in \mathcal{L}(V)$. A number $\lambda \in \mathbf{F}$ is called an *eigenvalue* of T if there exists $v \in V$ such that $v \neq 0$ and $Tv = \lambda v$.

5.7 *equivalent conditions to be an eigenvalue*

Suppose V is finite-dimensional, $T \in \mathcal{L}(V)$, and $\lambda \in \mathbf{F}$. Then the following are equivalent.

- (a) λ is an eigenvalue of T .
- (b) $T - \lambda I$ is not injective.
- (c) $T - \lambda I$ is not surjective.
- (d) $T - \lambda I$ is not invertible.

Reminder: $I \in \mathcal{L}(V)$ is the identity operator. Thus $Iv = v$ for all $v \in V$.

5.8 *definition: eigenvector*

Suppose $T \in \mathcal{L}(V)$ and $\lambda \in \mathbf{F}$ is an eigenvalue of T . A vector $v \in V$ is called an *eigenvector* of T corresponding to λ if $v \neq 0$ and $Tv = \lambda v$.

5.11 *linearly independent eigenvectors*

Suppose $T \in \mathcal{L}(V)$. Then every list of eigenvectors of T corresponding to distinct eigenvalues of T is linearly independent.

5.12 operator cannot have more eigenvalues than dimension of vector space

Suppose V is finite-dimensional. Then each operator on V has at most $\dim V$ distinct eigenvalues.

Polynomials Applied to Operators

5.13 notation: T^m

Suppose $T \in \mathcal{L}(V)$ and m is a positive integer.

- $T^m \in \mathcal{L}(V)$ is defined by $T^m = \underbrace{T \cdots T}_{m \text{ times}}$.
- T^0 is defined to be the identity operator I on V .
- If T is invertible with inverse T^{-1} , then $T^{-m} \in \mathcal{L}(V)$ is defined by

$$T^{-m} = (T^{-1})^m.$$

5.14 notation: $p(T)$

Suppose $T \in \mathcal{L}(V)$ and $p \in \mathcal{P}(\mathbf{F})$ is a polynomial given by

$$p(z) = a_0 + a_1z + a_2z^2 + \cdots + a_mz^m$$

for all $z \in \mathbf{F}$. Then $p(T)$ is the operator on V defined by

$$p(T) = a_0I + a_1T + a_2T^2 + \cdots + a_mT^m.$$

5.16 definition: *product of polynomials*

If $p, q \in \mathcal{P}(\mathbf{F})$, then $pq \in \mathcal{P}(\mathbf{F})$ is the polynomial defined by

$$(pq)(z) = p(z)q(z)$$

for all $z \in \mathbf{F}$.

5.17 *multiplicative properties*

Suppose $p, q \in \mathcal{P}(\mathbf{F})$ and $T \in \mathcal{L}(V)$.
Then

- (a) $(pq)(T) = p(T)q(T)$;
- (b) $p(T)q(T) = q(T)p(T)$.

Informal proof: When a product of polynomials is expanded using the distributive property, it does not matter whether the symbol is z or T .

5.18 *null space and range of $p(T)$ are invariant under T*

Suppose $T \in \mathcal{L}(V)$ and $p \in \mathcal{P}(\mathbf{F})$. Then $\text{null } p(T)$ and $\text{range } p(T)$ are invariant under T .