

3.44 notation: $A_{j,\cdot}$, $A_{\cdot,k}$

Suppose A is an m -by- n matrix.

- If $1 \leq j \leq m$, then $A_{j,\cdot}$ denotes the 1-by- n matrix consisting of row j of A .
- If $1 \leq k \leq n$, then $A_{\cdot,k}$ denotes the m -by-1 matrix consisting of column k of A .

3.46 *entry of matrix product equals row times column*

Suppose A is an m -by- n matrix and B is an n -by- p matrix. Then

$$(AB)_{j,k} = A_{j,\cdot} B_{\cdot,k}$$

if $1 \leq j \leq m$ and $1 \leq k \leq p$. In other words, the entry in row j , column k , of AB equals (row j of A) times (column k of B).

3.48 *column of matrix product equals matrix times column*

Suppose A is an m -by- n matrix and B is an n -by- p matrix. Then

$$(AB)_{\cdot,k} = AB_{\cdot,k}$$

if $1 \leq k \leq p$. In other words, column k of AB equals A times column k of B .

3.50 *linear combination of columns*

Suppose A is an m -by- n matrix and $b = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$ is an n -by-1 matrix. Then

$$Ab = b_1 A_{\cdot,1} + \cdots + b_n A_{\cdot,n}.$$

In other words, Ab is a linear combination of the columns of A , with the scalars that multiply the columns coming from b .

3.51 *matrix multiplication as linear combinations of columns*

Suppose C is an m -by- c matrix and R is a c -by- n matrix.

- (a) If $k \in \{1, \dots, n\}$, then column k of CR is a linear combination of the columns of C , with the coefficients of this linear combination coming from column k of R .
- (b) If $j \in \{1, \dots, m\}$, then row j of CR is a linear combination of the rows of R , with the coefficients of this linear combination coming from row j of C .

Column–Row Factorization and Rank of a Matrix

3.52 definition: *column rank*, *row rank*

Suppose A is an m -by- n matrix with entries in $\mathbf{F}$.

- The *column rank* of A is the dimension of the span of the columns of A in $\mathbf{F}^{m,1}$.
- The *row rank* of A is the dimension of the span of the rows of A in $\mathbf{F}^{1,n}$.

Ex

Suppose

$$A = \begin{pmatrix} 4 & 7 & 1 & 8 \\ 3 & 5 & 2 & 9 \end{pmatrix}.$$

The column rank of A is the dimension of

$$\text{span} \left(\begin{pmatrix} 4 \\ 3 \end{pmatrix}, \begin{pmatrix} 7 \\ 5 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 8 \\ 9 \end{pmatrix} \right)$$

The row rank of A is the dimension of

$$\text{span} \left((4 \ 7 \ 1 \ 8), (3 \ 5 \ 2 \ 9) \right)$$

3.54 definition: *transpose*, A^t

The *transpose* of a matrix A , denoted by A^t , is the matrix obtained from A by interchanging rows and columns. Specifically, if A is an m -by- n matrix, then A^t is the n -by- m matrix whose entries are given by the equation

$$(A^t)_{k,j} = A_{j,k}.$$

3.56 *column–row factorization*

Suppose A is an m -by- n matrix with entries in $\mathbf{F}$ and column rank $c \geq 1$. Then there exist an m -by- c matrix C and a c -by- n matrix R , both with entries in $\mathbf{F}$, such that $A = CR$.

3.57 *column rank equals row rank*

Suppose $A \in \mathbf{F}^{m,n}$. Then the column rank of A equals the row rank of A .

3.58 *definition: rank*

The *rank* of a matrix $A \in \mathbf{F}^{m,n}$ is the column rank of A .