3B Null Spaces and Ranges

Null Space and Injectivity

3.11 definition: *null space,* null T

For $T \in \mathcal{L}(V, W)$, the *null space* of T, denoted by null T, is the subset of V consisting of those vectors that T maps to 0:

$$\operatorname{null} T = \{v \in V : Tv = 0\}.$$

3.13 *the null space is a subspace*

Suppose $T \in \mathcal{L}(V, W)$. Then null T is a subspace of V.

3.14 definition: *injective*

A function $T\colon V \to W$ is called *injective* if $Tu = Tv$ implies $u = v$.

3.15 *injectivity* $\Longleftrightarrow$ *null space equals* $\{0\}$

Let $T \in \mathcal{L}(V, W)$. Then T is injective if and only if null $T = \{0\}$.

Range and Surjectivity

3.16 definition: *range*

For $T \in \mathcal{L}(V, W)$, the *range* of T is the subset of W consisting of those vectors that are equal to Tv for some $v \in V$:

$$\operatorname{range} T = \{Tv : v \in V\}.$$

3.18 *the range is a subspace*

If $T \in \mathcal{L}(V, W)$, then $\operatorname{range} T$ is a subspace of W.

3.19 definition: *surjective*

A function $T \colon V \to W$ is called *surjective* if its range equals W.

3.20 example: *surjectivity depends on the target space*

The differentiation map $D \in \mathcal{L}(\mathcal{P}_5(\mathbf{R}))$ defined by $Dp = p'$ is not surjective, because the polynomial x^5 is not in the range of D. However, the differentiation map $S \in \mathcal{L}(\mathcal{P}_5(\mathbf{R}), \mathcal{P}_4(\mathbf{R}))$ defined by $Sp = p'$ is surjective, because its range equals $\mathcal{P}_4(\mathbf{R})$, which is the vector space into which S maps.

Fundamental Theorem of Linear Maps

3.21 *fundamental theorem of linear maps*

Suppose V is finite-dimensional and $T \in \mathcal{L}(V, W)$. Then range T is finite-dimensional and

$$\dim V = \dim \operatorname{null} T + \dim \operatorname{range} T.$$

3.22 *linear map to a lower-dimensional space is not injective*

Suppose V and W are finite-dimensional vector spaces such that $\dim V > \dim W$. Then no linear map from V to W is injective.

3.24 *linear map to a higher-dimensional space is not surjective*

Suppose V and W are finite-dimensional vector spaces such that $\dim V < \dim W$. Then no linear map from V to W is surjective.

3.26 *homogeneous system of linear equations*

A homogeneous system of linear equations with more variables than equations has nonzero solutions.

3.28 *inhomogeneous system of linear equations*

An inhomogeneous system of linear equations with more equations than variables has no solution for some choice of the constant terms.