2A Span and Linear Independence

2.2 definition: *linear combination*

A *linear combination* of a list $v_1, \ldots, v_m$ of vectors in V is a vector of the form

$$a_1 v_1 + \cdots + a_m v_m,$$

where $a_1, \ldots, a_m \in \mathbf{F}$.

2.4 definition: *span*

The set of all linear combinations of a list of vectors $v_1, \ldots, v_m$ in V is called the *span* of $v_1, \ldots, v_m$, denoted by $\operatorname{span}(v_1, \ldots, v_m)$. In other words,

$$\operatorname{span}(v_1, \ldots, v_m) = \{a_1 v_1 + \cdots + a_m v_m : a_1, \ldots, a_m \in \mathbf{F}\}.$$

The span of the empty list $(\ )$ is defined to be $\{0\}$.

2.6 *span is the smallest containing subspace*

The span of a list of vectors in V is the smallest subspace of V containing all vectors in the list.

2.7 definition: *spans*

If $\operatorname{span}(v_1, \dots, v_m)$ equals V, we say that the list $v_1, \dots, v_m$ *spans* V.

2.9 definition: *finite-dimensional vector space*

A vector space is called *finite-dimensional* if some list of vectors in it spans the space.

Recall that by definition every list has finite length.

2.13 definition: *infinite-dimensional vector space*

A vector space is called *infinite-dimensional* if it is not finite-dimensional.

2.10 definition: *polynomial,* $\mathcal{P}(\mathbf{F})$

- A function $p\colon \mathbf{F} \to \mathbf{F}$ is called a *polynomial* with coefficients in $\mathbf{F}$ if there exist $a_0, \dots, a_m \in \mathbf{F}$ such that

$$p(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_m z^m$$

  for all $z \in \mathbf{F}$.
- $\mathcal{P}(\mathbf{F})$ is the set of all polynomials with coefficients in $\mathbf{F}$.

2.12 notation: $\mathcal{P}_m(\mathbf{F})$

For m a nonnegative integer, $\mathcal{P}_m(\mathbf{F})$ denotes the set of all polynomials with coefficients in $\mathbf{F}$ and degree at most m.

2.15 definition: *linearly independent*

- A list $v_1, \ldots, v_m$ of vectors in V is called *linearly independent* if the only choice of $a_1, \ldots, a_m \in \mathbf{F}$ that makes

$$a_1 v_1 + \cdots + a_m v_m = 0$$

  is $a_1 = \cdots = a_m = 0$.
- The empty list $(\,)$ is also declared to be linearly independent.

2.17 definition: *linearly dependent*

- A list of vectors in V is called *linearly dependent* if it is not linearly independent.
- In other words, a list $v_1, \ldots, v_m$ of vectors in V is linearly dependent if there exist $a_1, \ldots, a_m \in \mathbf{F}$, not all 0, such that $a_1 v_1 + \cdots + a_m v_m = 0$.

2.19 *linear dependence lemma*

Suppose $v_1, \ldots, v_m$ is a linearly dependent list in V. Then there exists $k \in \{1, 2, \ldots, m\}$ such that

$$v_k \in \operatorname{span}(v_1, \ldots, v_{k-1}).$$

Furthermore, if k satisfies the condition above and the k^{th} term is removed from $v_1, \ldots, v_m$, then the span of the remaining list equals $\operatorname{span}(v_1, \ldots, v_m)$.

2.22 *length of linearly independent list $\leq$ length of spanning list*

In a finite-dimensional vector space, the length of every linearly independent list of vectors is less than or equal to the length of every spanning list of vectors.

2.25 *finite-dimensional subspaces*

Every subspace of a finite-dimensional vector space is finite-dimensional.