1C Subspaces

1.33 definition: *subspace*

A subset U of V is called a *subspace* of V if U is also a vector space with the same additive identity, addition, and scalar multiplication as on V.

1.34 *conditions for a subspace*

A subset U of V is a subspace of V if and only if U satisfies the following three conditions.

additive identity

$0 \in U$.

closed under addition

$u, w \in U$ implies $u + w \in U$.

closed under scalar multiplication

$a \in \mathbf{F}$ and $u \in U$ implies $au \in U$.

1.36 definition: *sum of subspaces*

Suppose $V_1, \ldots, V_m$ are subspaces of V. The *sum* of $V_1, \ldots, V_m$, denoted by $V_1 + \cdots + V_m$, is the set of all possible sums of elements of $V_1, \ldots, V_m$. More precisely,

$$V_1 + \cdots + V_m = \{v_1 + \cdots + v_m : v_1 \in V_1, \ldots, v_m \in V_m\}.$$

1.40 *sum of subspaces is the smallest containing subspace*

Suppose $V_1, \ldots, V_m$ are subspaces of V. Then $V_1 + \cdots + V_m$ is the smallest subspace of V containing $V_1, \ldots, V_m$.

1.41 definition: *direct sum,* $\oplus$

Suppose $V_1, \ldots, V_m$ are subspaces of V.

- The sum $V_1 + \cdots + V_m$ is called a *direct sum* if each element of $V_1 + \cdots + V_m$ can be written in only one way as a sum $v_1 + \cdots + v_m$, where each $v_k \in V_k$.
- If $V_1 + \cdots + V_m$ is a direct sum, then $V_1 \oplus \cdots \oplus V_m$ denotes $V_1 + \cdots + V_m$, with the $\oplus$ notation serving as an indication that this is a direct sum.

1.45 *condition for a direct sum*

Suppose $V_1, \ldots, V_m$ are subspaces of V. Then $V_1 + \cdots + V_m$ is a direct sum if and only if the only way to write 0 as a sum $v_1 + \cdots + v_m$, where each $v_k \in V_k$, is by taking each v_k equal to 0.

1.46 *direct sum of two subspaces*

Suppose U and W are subspaces of V. Then

$$U + W \text{ is a direct sum} \iff U \cap W = \{0\}.$$