7D Isometries, Unitary Operators, and Matrix Factorization

Isometries

7.44 definition: *isometry*

A linear map $S \in \mathcal{L}(V, W)$ is called an *isometry* if

$$\|Sv\| = \|v\|$$

for every $v \in V$. In other words, a linear map is an isometry if it preserves norms.

7.49 *characterization of isometries*

Suppose $S \in \mathcal{L}(V, W)$. Suppose $e_1, \ldots, e_n$ is an orthonormal basis of V and $f_1, \ldots, f_m$ is an orthonormal basis of W. Then the following are equivalent.

(a) S is an isometry.

(b) $S^*S = I$.

(c) $\langle Su, Sv \rangle = \langle u, v \rangle$ for all $u, v \in V$.

(d) $Se_1, \ldots, Se_n$ is an orthonormal list in W.

(e) The columns of $\mathcal{M}(S, (e_1, \ldots, e_n), (f_1, \ldots, f_m))$ form an orthonormal list in $\mathbf{F}^m$ with respect to the Euclidean inner product.

Unitary Operators

7.51 definition: *unitary operator*

An operator $S \in \mathcal{L}(V)$ is called *unitary* if S is an invertible isometry.

7.53 *characterization of unitary operators*

Suppose $S \in \mathcal{L}(V)$. Suppose $e_1, \dots, e_n$ is an orthonormal basis of V. Then the following are equivalent.

(a) S is a unitary operator.

(b) $S^*S = SS^* = I$.

(c) S is invertible and $S^{-1} = S^*$.

(d) $Se_1, \dots, Se_n$ is an orthonormal basis of V.

(e) The rows of $\mathcal{M}(S, (e_1, \dots, e_n))$ form an orthonormal basis of $\mathbf{F}^n$ with respect to the Euclidean inner product.

(f) S^* is a unitary operator.

7.55 *description of unitary operators on complex inner product spaces*

Suppose $\mathbf{F} = \mathbf{C}$ and $S \in \mathcal{L}(V)$. Then the following are equivalent.

(a) S is a unitary operator.

(b) There is an orthonormal basis of V consisting of eigenvectors of S whose corresponding eigenvalues all have absolute value 1.

QR Factorization

7.56 definition: *unitary matrix*

An n-by-n matrix is called *unitary* if its columns form an orthonormal list in $\mathbf{F}^n$.

7.57 *characterizations of unitary matrices*

Suppose Q is an n-by-n matrix. Then the following are equivalent.

(a) Q is a unitary matrix.

(b) The rows of Q form an orthonormal list in $\mathbf{F}^n$.

(c) $\|Qv\| = \|v\|$ for every $v \in \mathbf{F}^n$.

(d) $Q^*Q = QQ^* = I$, the n-by-n matrix with 1's on the diagonal and 0's elsewhere.

7.58 *QR factorization*

Suppose A is a square matrix with linearly independent columns. Then there exist unique matrices Q and R such that Q is unitary, R is upper triangular with only positive numbers on its diagonal, and

$$A = QR.$$