

7.14 $\langle Tv, v \rangle$ is real for all $v \iff T$ is self-adjoint (assuming $\mathbf{F} = \mathbf{C}$)

Suppose V is a complex inner product space and $T \in \mathcal{L}(V)$. Then

T is self-adjoint $\iff \langle Tv, v \rangle \in \mathbf{R}$ for every $v \in V$.

7.16 T self-adjoint and $\langle Tv, v \rangle = 0$ for all $v \iff T = 0$

Suppose T is a self-adjoint operator on V . Then

$\langle Tv, v \rangle = 0$ for every $v \in V \iff T = 0$.

Normal Operators

7.18 definition: *normal*

- An operator on an inner product space is called *normal* if it commutes with its adjoint.
- In other words, $T \in \mathcal{L}(V)$ is normal if $TT^* = T^*T$.

7.20 T is normal if and only if Tv and T^*v have the same norm

Suppose $T \in \mathcal{L}(V)$. Then

$$T \text{ is normal} \iff \|Tv\| = \|T^*v\| \text{ for every } v \in V.$$

7.21 *range, null space, and eigenvectors of a normal operator*

Suppose $T \in \mathcal{L}(V)$ is normal. Then

- (a) $\text{null } T = \text{null } T^*$;
- (b) $\text{range } T = \text{range } T^*$;
- (c) $V = \text{null } T \oplus \text{range } T$;
- (d) $T - \lambda I$ is normal for every $\lambda \in \mathbf{F}$;
- (e) if $v \in V$ and $\lambda \in \mathbf{F}$, then $Tv = \lambda v$ if and only if $T^*v = \overline{\lambda}v$.

7.22 *orthogonal eigenvectors for normal operators*

Suppose $T \in \mathcal{L}(V)$ is normal. Then eigenvectors of T corresponding to distinct eigenvalues are orthogonal.

7.23 *T is normal $\iff$ the real and imaginary parts of T commute*

Suppose $\mathbf{F} = \mathbf{C}$ and $T \in \mathcal{L}(V)$. Then T is normal if and only if there exist commuting self-adjoint operators A and B such that $T = A + iB$.