6C Orthogonal Complements and Minimization Problems

6.46 definition: *orthogonal complement, $U^\perp$*

If U is a subset of V, then the *orthogonal complement* of U, denoted by $U^\perp$, is the set of all vectors in V that are orthogonal to every vector in U:

$$U^\perp = \{v \in V : \langle u, v \rangle = 0 \text{ for every } u \in U\}.$$

6.48 *properties of orthogonal complement*

(a) If U is a subset of V, then $U^\perp$ is a subspace of V.

(b) $\{0\}^\perp = V$.

(c) $V^\perp = \{0\}$.

(d) If U is a subset of V, then $U \cap U^\perp \subseteq \{0\}$.

(e) If G and H are subsets of V and $G \subseteq H$, then $H^\perp \subseteq G^\perp$.

6.49 *direct sum of a subspace and its orthogonal complement*

Suppose U is a finite-dimensional subspace of V. Then

$$V = U \oplus U^{\perp}.$$

6.51 *dimension of orthogonal complement*

Suppose V is finite-dimensional and U is a subspace of V. Then

$$\dim U^{\perp} = \dim V - \dim U.$$

6.52 *orthogonal complement of the orthogonal complement*

Suppose U is a finite-dimensional subspace of V. Then

$$U = (U^{\perp})^{\perp}.$$

6.54 $U^{\perp} = \{0\} \iff U = V$ *(for U a finite-dimensional subspace of V)*

Suppose U is a finite-dimensional subspace of V. Then

$$U^{\perp} = \{0\} \iff U = V.$$

6.55 definition: *orthogonal projection, P_U*

Suppose U is a finite-dimensional subspace of V. The *orthogonal projection* of V onto U is the operator $P_U \in \mathcal{L}(V)$ defined as follows: For each $v \in V$, write $v = u + w$, where $u \in U$ and $w \in U^\perp$. Then let $P_U v = u$.

6.57 *properties of orthogonal projection P_U*

Suppose U is a finite-dimensional subspace of V. Then

(a) $P_U \in \mathcal{L}(V)$;

(b) $P_U u = u$ for every $u \in U$;

(c) $P_U w = 0$ for every $w \in U^\perp$;

(d) $\operatorname{range} P_U = U$;

(e) $\operatorname{null} P_U = U^\perp$;

(f) $v - P_U v \in U^\perp$ for every $v \in V$;

(g) $P_U^2 = P_U$;

(h) $\|P_U v\| \leq \|v\|$ for every $v \in V$;

(i) if $e_1, \ldots, e_m$ is an orthonormal basis of U and $v \in V$, then

$$P_U v = \langle v, e_1 \rangle e_1 + \cdots + \langle v, e_m \rangle e_m.$$

6.58 *Riesz representation theorem, revisited*

Suppose V is finite-dimensional. For each $v \in V$, define $\varphi_v \in V'$ by

$$\varphi_v(u) = \langle u, v \rangle$$

for each $u \in V$. Then $v \mapsto \varphi_v$ is a one-to-one function from V onto V'.