

6A Inner Products and Norms

6.1 definition: *dot product*

For $x, y \in \mathbf{R}^n$, the *dot product* of x and y , denoted by $x \cdot y$, is defined by

$$x \cdot y = x_1 y_1 + \cdots + x_n y_n,$$

where $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$.

6.2 definition: *inner product*

An *inner product* on V is a function that takes each ordered pair (u, v) of elements of V to a number $\langle u, v \rangle \in \mathbf{F}$ and has the following properties.

positivity

$$\langle v, v \rangle \geq 0 \text{ for all } v \in V.$$

definiteness

$$\langle v, v \rangle = 0 \text{ if and only if } v = 0.$$

additivity in first slot

$$\langle u + v, w \rangle = \langle u, w \rangle + \langle v, w \rangle \text{ for all } u, v, w \in V.$$

homogeneity in first slot

$$\langle \lambda u, v \rangle = \lambda \langle u, v \rangle \text{ for all } \lambda \in \mathbf{F} \text{ and all } u, v \in V.$$

conjugate symmetry

$$\langle u, v \rangle = \overline{\langle v, u \rangle} \text{ for all } u, v \in V.$$

6.4 definition: *inner product space*

An *inner product space* is a vector space V along with an inner product on V .

6.6 *basic properties of an inner product*

- (a) For each fixed $v \in V$, the function that takes $u \in V$ to $\langle u, v \rangle$ is a linear map from V to $\mathbf{F}$.
- (b) $\langle 0, v \rangle = 0$ for every $v \in V$.
- (c) $\langle v, 0 \rangle = 0$ for every $v \in V$.
- (d) $\langle u, v + w \rangle = \langle u, v \rangle + \langle u, w \rangle$ for all $u, v, w \in V$.
- (e) $\langle u, \lambda v \rangle = \overline{\lambda} \langle u, v \rangle$ for all $\lambda \in \mathbf{F}$ and all $u, v \in V$.

6.7 definition: *norm*, $\|v\|$

For $v \in V$, the *norm* of v , denoted by $\|v\|$, is defined by

$$\|v\| = \sqrt{\langle v, v \rangle}.$$

6.9 *basic properties of the norm*

Suppose $v \in V$.

- (a) $\|v\| = 0$ if and only if $v = 0$.
- (b) $\|\lambda v\| = |\lambda| \|v\|$ for all $\lambda \in \mathbb{F}$.

6.10 definition: *orthogonal*

Two vectors $u, v \in V$ are called *orthogonal* if $\langle u, v \rangle = 0$.

6.11 *orthogonality and 0*

- (a) 0 is orthogonal to every vector in V .
- (b) 0 is the only vector in V that is orthogonal to itself.

6.12 *Pythagorean theorem*

Suppose $u, v \in V$. If u and v are orthogonal, then

$$\|u + v\|^2 = \|u\|^2 + \|v\|^2.$$

6.13 *an orthogonal decomposition*

Suppose $u, v \in V$, with $v \neq 0$. Set $c = \frac{\langle u, v \rangle}{\|v\|^2}$ and $w = u - \frac{\langle u, v \rangle}{\|v\|^2}v$. Then

$$u = cv + w \quad \text{and} \quad \langle w, v \rangle = 0.$$

6.14 *Cauchy–Schwarz inequality*

Suppose $u, v \in V$. Then

$$|\langle u, v \rangle| \leq \|u\| \|v\|.$$

This inequality is an equality if and only if one of u, v is a scalar multiple of the other.

6.17 *triangle inequality*

Suppose $u, v \in V$. Then

$$\|u + v\| \leq \|u\| + \|v\|.$$

This inequality is an equality if and only if one of u, v is a nonnegative real multiple of the other.

6.21 *parallelogram equality*

Suppose $u, v \in V$. Then

$$\|u + v\|^2 + \|u - v\|^2 = 2(\|u\|^2 + \|v\|^2).$$