5C Upper-Triangular Matrices

5.35 definition: *matrix of an operator, $\mathcal{M}(T)$*

Suppose $T \in \mathcal{L}(V)$. The *matrix of* T with respect to a basis $v_1, \ldots, v_n$ of V is the n-by-n matrix

$$\mathcal{M}(T) = \begin{pmatrix} A_{1,1} & \cdots & A_{1,n} \\ \vdots & & \vdots \\ A_{n,1} & \cdots & A_{n,n} \end{pmatrix}$$

whose entries $A_{j,k}$ are defined by

$$Tv_k = A_{1,k}v_1 + \cdots + A_{n,k}v_n.$$

The notation $\mathcal{M}(T, (v_1, \ldots, v_n))$ is used if the basis is not clear from the context.

5.37 definition: *diagonal of a matrix*

The *diagonal* of a square matrix consists of the entries on the line from the upper left corner to the bottom right corner.

5.38 definition: ***upper-triangular matrix***

A square matrix is called *upper triangular* if all entries below the diagonal are 0.

5.39 ***conditions for upper-triangular matrix***

Suppose $T \in \mathcal{L}(V)$ and $v_1, \ldots, v_n$ is a basis of V. Then the following are equivalent.

(a) The matrix of T with respect to $v_1, \ldots, v_n$ is upper triangular.

(b) $\operatorname{span}(v_1, \ldots, v_k)$ is invariant under T for each $k = 1, \ldots, n$.

(c) $Tv_k \in \operatorname{span}(v_1, \ldots, v_k)$ for each $k = 1, \ldots, n$.

5.40 *equation satisfied by operator with upper-triangular matrix*

Suppose $T \in \mathcal{L}(V)$ and V has a basis with respect to which T has an upper-triangular matrix with diagonal entries $\lambda_1, \dots, \lambda_n$. Then

$$(T - \lambda_1 I) \cdots (T - \lambda_n I) = 0.$$