

3F Duality

Dual Space and Dual Map

3.108 definition: *linear functional*

A *linear functional* on V is a linear map from V to $\mathbf{F}$. In other words, a linear functional is an element of $\mathcal{L}(V, \mathbf{F})$.

3.110 definition: *dual space, V'*

The *dual space* of V , denoted by V' , is the vector space of all linear functionals on V . In other words, $V' = \mathcal{L}(V, \mathbf{F})$.

3.111 $\dim V' = \dim V$

Suppose V is finite-dimensional. Then V' is also finite-dimensional and
$$\dim V' = \dim V.$$

3.112 definition: *dual basis*

If $v_1, \dots, v_n$ is a basis of V , then the *dual basis* of $v_1, \dots, v_n$ is the list $\varphi_1, \dots, \varphi_n$ of elements of V' , where each φ_j is the linear functional on V such that

$$\varphi_j(v_k) = \begin{cases} 1 & \text{if } k = j, \\ 0 & \text{if } k \neq j. \end{cases}$$

3.114 *dual basis gives coefficients for linear combination*

Suppose $v_1, \dots, v_n$ is a basis of V and $\varphi_1, \dots, \varphi_n$ is the dual basis. Then

$$v = \varphi_1(v)v_1 + \cdots + \varphi_n(v)v_n$$

for each $v \in V$.

3.116 *dual basis is a basis of the dual space*

Suppose V is finite-dimensional. Then the dual basis of a basis of V is a basis of V' .

3.118 definition: *dual map*, T'

Suppose $T \in \mathcal{L}(V, W)$. The *dual map* of T is the linear map $T' \in \mathcal{L}(W', V')$ defined for each $\varphi \in W'$ by

$$T'(\varphi) = \varphi \circ T.$$

3.120 *algebraic properties of dual maps*

Suppose $T \in \mathcal{L}(V, W)$. Then

- (a) $(S + T)' = S' + T'$ for all $S \in \mathcal{L}(V, W)$;
- (b) $(\lambda T)' = \lambda T'$ for all $\lambda \in \mathbf{F}$;
- (c) $(ST)' = T'S'$ for all $S \in \mathcal{L}(W, U)$.

Null Space and Range of Dual of Linear Map

3.121 definition: *annihilator*, U^0

For $U \subseteq V$, the *annihilator* of U , denoted by U^0 , is defined by

$$U^0 = \{\varphi \in V' : \varphi(u) = 0 \text{ for all } u \in U\}.$$

3.124 *the annihilator is a subspace*

Suppose $U \subseteq V$. Then U^0 is a subspace of V' .