3E Products and Quotients of Vector Spaces

Products of Vector Spaces

3.87 definition: *product of vector spaces*

Suppose $V_1, \dots, V_m$ are vector spaces over $\mathbf{F}$.

- The *product* $V_1 \times \cdots \times V_m$ is defined by

$$V_1 \times \cdots \times V_m = \{(v_1, \dots, v_m) : v_1 \in V_1, \dots, v_m \in V_m\}.$$

- Addition on $V_1 \times \cdots \times V_m$ is defined by

$$(u_1, \dots, u_m) + (v_1, \dots, v_m) = (u_1 + v_1, \dots, u_m + v_m).$$

- Scalar multiplication on $V_1 \times \cdots \times V_m$ is defined by

$$\lambda(v_1, \dots, v_m) = (\lambda v_1, \dots, \lambda v_m).$$

3.89 *product of vector spaces is a vector space*

Suppose $V_1, \dots, V_m$ are vector spaces over $\mathbf{F}$. Then $V_1 \times \cdots \times V_m$ is a vector space over $\mathbf{F}$.

3.92 *dimension of a product is the sum of dimensions*

Suppose $V_1, \dots, V_m$ are finite-dimensional vector spaces. Then $V_1 \times \cdots \times V_m$ is finite-dimensional and

$$\dim(V_1 \times \cdots \times V_m) = \dim V_1 + \cdots + \dim V_m.$$

3.93 *products and direct sums*

Suppose that $V_1, \ldots, V_m$ are subspaces of V. Define a linear map $\Gamma : V_1 \times \cdots \times V_m \to V_1 + \cdots + V_m$ by

$$\Gamma(v_1, \ldots, v_m) = v_1 + \cdots + v_m.$$

Then $V_1 + \cdots + V_m$ is a direct sum if and only if Γ is injective.

3.94 *a sum is a direct sum if and only if dimensions add up*

Suppose V is finite-dimensional and $V_1, \ldots, V_m$ are subspaces of V. Then $V_1 + \cdots + V_m$ is a direct sum if and only if

$$\dim(V_1 + \cdots + V_m) = \dim V_1 + \cdots + \dim V_m.$$

Quotient Spaces

3.95 notation: $v + U$

Suppose $v \in V$ and $U \subseteq V$. Then $v + U$ is the subset of V defined by

$$v + U = \{v + u : u \in U\}.$$

3.97 definition: *translate*

For $v \in V$ and U a subset of V, the set $v + U$ is said to be a *translate* of U.

3.99 definition: *quotient space, V/U*

Suppose U is a subspace of V. Then the *quotient space* V/U is the set of all translates of U. Thus

$$V/U = \{v + U : v \in V\}.$$

3.101 *two translates of a subspace are equal or disjoint*

Suppose U is a subspace of V and $v, w \in V$. Then

$$v - w \in U \iff v + U = w + U \iff (v + U) \cap (w + U) \neq \emptyset.$$

3.102 definition: *addition and scalar multiplication on V/U*

Suppose U is a subspace of V. Then *addition* and *scalar multiplication* are defined on V/U by

$$(v+U)+(w+U)=(v+w)+U$$
$$\lambda(v+U)=(\lambda v)+U$$

for all $v,w\in V$ and all $\lambda\in\mathbf{F}$.

3.103 *quotient space is a vector space*

Suppose U is a subspace of V. Then V/U, with the operations of addition and scalar multiplication as defined above, is a vector space.

3.104 definition: *quotient map, π*

Suppose U is a subspace of V. The *quotient map* $\pi\colon V \to V/U$ is the linear map defined by

$$\pi(v) = v + U$$

for each $v \in V$.

3.105 *dimension of quotient space*

Suppose V is finite-dimensional and U is a subspace of V. Then

$$\dim V/U = \dim V - \dim U.$$

3.106 notation: $\widetilde{T}$

Suppose $T \in \mathcal{L}(V, W)$. Define $\widetilde{T} \colon V/(\operatorname{null} T) \to W$ by

$$\widetilde{T}(v + \operatorname{null} T) = Tv.$$

3.107 ***null space and range of $\widetilde{T}$***

Suppose $T \in \mathcal{L}(V, W)$. Then

(a) $\widetilde{T} \circ \pi = T$, where π is the quotient map of V onto $V/(\operatorname{null} T)$;

(b) $\widetilde{T}$ is injective;

(c) $\operatorname{range} \widetilde{T} = \operatorname{range} T$;

(d) $V/(\operatorname{null} T)$ and $\operatorname{range} T$ are isomorphic vector spaces.