3D Invertibility and Isomorphisms

Invertible Linear Maps

3.59 definition: *invertible, inverse*

- A linear map $T \in \mathcal{L}(V, W)$ is called *invertible* if there exists a linear map $S \in \mathcal{L}(W, V)$ such that ST equals the identity operator on V and TS equals the identity operator on W.
- A linear map $S \in \mathcal{L}(W, V)$ satisfying $ST = I$ and $TS = I$ is called an *inverse* of T (note that the first I is the identity operator on V and the second I is the identity operator on W).

3.60 *inverse is unique*

An invertible linear map has a unique inverse.

3.61 notation: T^{-1}

If T is invertible, then its inverse is denoted by T^{-1}. In other words, if $T \in \mathcal{L}(V, W)$ is invertible, then T^{-1} is the unique element of $\mathcal{L}(W, V)$ such that $T^{-1}T = I$ and $TT^{-1} = I$.

3.63 *invertibility* $\iff$ *injectivity and surjectivity*

A linear map is invertible if and only if it is injective and surjective.

3.65 *injectivity is equivalent to surjectivity (if* $\dim V = \dim W < \infty$*)*

Suppose that V and W are finite-dimensional vector spaces, $\dim V = \dim W$, and $T \in \mathcal{L}(V, W)$. Then

$$T \text{ is invertible} \iff T \text{ is injective} \iff T \text{ is surjective.}$$

$\boxed{\text{Ex}}$ Define $T \in \mathcal{L}(\mathcal{P}_m(\mathbb{R}))$ by $(Tp)(x) = ((x^2+5x+7)p(x))''$.

3.68 $ST = I \iff TS = I$ *(on vector spaces of the same dimension)*

Suppose V and W are finite-dimensional vector spaces of the same dimension, $S \in \mathcal{L}(V, W)$, and $T \in \mathcal{L}(W, V)$. Then $ST = I$ if and only if $TS = I$.

Isomorphic Vector Spaces

3.69 definition: *isomorphism, isomorphic*

- An *isomorphism* is an invertible linear map.
- Two vector spaces are called *isomorphic* if there is an isomorphism from one vector space onto the other one.

3.70 *dimension shows whether vector spaces are isomorphic*

Two finite-dimensional vector spaces over $\mathbf{F}$ are isomorphic if and only if they have the same dimension.