3.44 notation: $A_{j,\cdot}$, $A_{\cdot,k}$

Suppose A is an m-by-n matrix.

- If $1 \leq j \leq m$, then $A_{j,\cdot}$ denotes the 1-by-n matrix consisting of row j of A.
- If $1 \leq k \leq n$, then $A_{\cdot,k}$ denotes the m-by-1 matrix consisting of column k of A.

3.46 *entry of matrix product equals row times column*

Suppose A is an m-by-n matrix and B is an n-by-p matrix. Then

$$(AB)_{j,k} = A_{j,\cdot} B_{\cdot,k}$$

if $1 \leq j \leq m$ and $1 \leq k \leq p$. In other words, the entry in row j, column k, of AB equals (row j of A) times (column k of B).

3.48 *column of matrix product equals matrix times column*

Suppose A is an m-by-n matrix and B is an n-by-p matrix. Then

$$(AB)_{\cdot,k} = AB_{\cdot,k}$$

if $1 \leq k \leq p$. In other words, column k of AB equals A times column k of B.

3.50 *linear combination of columns*

Suppose A is an m-by-n matrix and $b = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$ is an n-by-1 matrix. Then

$$Ab = b_1 A_{\cdot,1} + \cdots + b_n A_{\cdot,n}.$$

In other words, Ab is a linear combination of the columns of A, with the scalars that multiply the columns coming from b.

3.51 *matrix multiplication as linear combinations of columns*

Suppose C is an m-by-c matrix and R is a c-by-n matrix.

(a) If $k \in \{1, \dots, n\}$, then column k of CR is a linear combination of the columns of C, with the coefficients of this linear combination coming from column k of R.

(b) If $j \in \{1, \dots, m\}$, then row j of CR is a linear combination of the rows of R, with the coefficients of this linear combination coming from row j of C.

3.52 definition: *column rank, row rank*

Suppose A is an m-by-n matrix with entries in $\mathbf{F}$.

- The *column rank* of A is the dimension of the span of the columns of A in $\mathbf{F}^{m,1}$.
- The *row rank* of A is the dimension of the span of the rows of A in $\mathbf{F}^{1,n}$.

$\boxed{\text{Ex}}$

Suppose

$$A = \begin{pmatrix} 4 & 7 & 1 & 8 \\ 3 & 5 & 2 & 9 \end{pmatrix}.$$

The column rank of A is the dimension of

$$\operatorname{span}\left(\begin{pmatrix} 4 \\ 3 \end{pmatrix}, \begin{pmatrix} 7 \\ 5 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 8 \\ 9 \end{pmatrix}\right)$$

The row rank of A is the dimension of

$$\operatorname{span}\big(\begin{pmatrix} 4 & 7 & 1 & 8 \end{pmatrix}, \begin{pmatrix} 3 & 5 & 2 & 9 \end{pmatrix}\big)$$

3.54 definition: *transpose,* A^{t}

The *transpose* of a matrix A, denoted by A^{t}, is the matrix obtained from A by interchanging rows and columns. Specifically, if A is an m-by-n matrix, then A^{t} is the n-by-m matrix whose entries are given by the equation

$$(A^{\mathrm{t}})_{k,j} = A_{j,k}.$$

3.56 *column–row factorization*

Suppose A is an m-by-n matrix with entries in $\mathbf{F}$ and column rank $c \geq 1$. Then there exist an m-by-c matrix C and a c-by-n matrix R, both with entries in $\mathbf{F}$, such that $A = CR$.

3.57 *column rank equals row rank*

Suppose $A \in \mathbf{F}^{m,n}$. Then the column rank of A equals the row rank of A.

3.58 definition: *rank*

The *rank* of a matrix $A \in \mathbf{F}^{m,n}$ is the column rank of A.