

3C Matrices

Representing a Linear Map by a Matrix

3.29 definition: *matrix*, $A_{j,k}$

Suppose m and n are nonnegative integers. An m -by- n *matrix* A is a rectangular array of elements of $\mathbf{F}$ with m rows and n columns:

$$A = \begin{pmatrix} A_{1,1} & \cdots & A_{1,n} \\ \vdots & & \vdots \\ A_{m,1} & \cdots & A_{m,n} \end{pmatrix}.$$

The notation $A_{j,k}$ denotes the entry in row j , column k of A .

3.31 definition: *matrix of a linear map*, $\mathcal{M}(T)$

Suppose $T \in \mathcal{L}(V, W)$ and $v_1, \dots, v_n$ is a basis of V and $w_1, \dots, w_m$ is a basis of W . The *matrix of T* with respect to these bases is the m -by- n matrix $\mathcal{M}(T)$ whose entries $A_{j,k}$ are defined by

$$Tv_k = A_{1,k}w_1 + \cdots + A_{m,k}w_m.$$

If the bases $v_1, \dots, v_n$ and $w_1, \dots, w_m$ are not clear from the context, then the notation $\mathcal{M}(T, (v_1, \dots, v_n), (w_1, \dots, w_m))$ is used.

Idea

$$\mathcal{M}(T) = \begin{matrix} & v_1 & \cdots & v_k & \cdots & v_n \\ \begin{matrix} w_1 \\ \vdots \\ w_m \end{matrix} & \begin{pmatrix} A_{1,k} \\ \vdots \\ A_{m,k} \end{pmatrix} \end{matrix}.$$

The k^{th} column of $\mathcal{M}(T)$ consists of the scalars needed to write Tv_k as a linear combination of $w_1, \dots, w_m$:

$$Tv_k = \sum_{j=1}^m A_{j,k}w_j.$$

EX

Suppose $T \in \mathcal{L}(\mathbf{F}^2, \mathbf{F}^3)$ is defined by

$$T(x, y) = (x + 3y, 2x + 5y, 7x + 9y).$$

EX

Suppose $D \in \mathcal{L}(\mathcal{P}_3(\mathbf{R}), \mathcal{P}_2(\mathbf{R}))$ is the differentiation map defined by $Dp = p'$.

Addition and Scalar Multiplication of Matrices

3.34 definition: *matrix addition*

The *sum of two matrices of the same size* is the matrix obtained by adding corresponding entries in the matrices:

$$\begin{pmatrix} A_{1,1} & \cdots & A_{1,n} \\ \vdots & & \vdots \\ A_{m,1} & \cdots & A_{m,n} \end{pmatrix} + \begin{pmatrix} C_{1,1} & \cdots & C_{1,n} \\ \vdots & & \vdots \\ C_{m,1} & \cdots & C_{m,n} \end{pmatrix} = \begin{pmatrix} A_{1,1} + C_{1,1} & \cdots & A_{1,n} + C_{1,n} \\ \vdots & & \vdots \\ A_{m,1} + C_{m,1} & \cdots & A_{m,n} + C_{m,n} \end{pmatrix}.$$

3.35 *matrix of the sum of linear maps*

Suppose $S, T \in \mathcal{L}(V, W)$. Then $\mathcal{M}(S + T) = \mathcal{M}(S) + \mathcal{M}(T)$.

3.36 definition: *scalar multiplication of a matrix*

The product of a scalar and a matrix is the matrix obtained by multiplying each entry in the matrix by the scalar:

$$\lambda \begin{pmatrix} A_{1,1} & \cdots & A_{1,n} \\ \vdots & & \vdots \\ A_{m,1} & \cdots & A_{m,n} \end{pmatrix} = \begin{pmatrix} \lambda A_{1,1} & \cdots & \lambda A_{1,n} \\ \vdots & & \vdots \\ \lambda A_{m,1} & \cdots & \lambda A_{m,n} \end{pmatrix}.$$

3.38 *the matrix of a scalar times a linear map*

Suppose $\lambda \in \mathbf{F}$ and $T \in \mathcal{L}(V, W)$. Then $\mathcal{M}(\lambda T) = \lambda \mathcal{M}(T)$.

3.39 notation: $\mathbf{F}^{m,n}$

For m and n positive integers, the set of all m -by- n matrices with entries in $\mathbf{F}$ is denoted by $\mathbf{F}^{m,n}$.

3.40 $\dim \mathbf{F}^{m,n} = mn$

Suppose m and n are positive integers. With addition and scalar multiplication defined as above, $\mathbf{F}^{m,n}$ is a vector space of dimension mn .

Matrix Multiplication

Consider linear maps $T: U \rightarrow V$ and $S: V \rightarrow W$. The composition ST is a linear map from U to W . Does $\mathcal{M}(ST)$ equal $\mathcal{M}(S)\mathcal{M}(T)$? This question does not yet make sense because we have not defined the product of two matrices. We will choose a definition of matrix multiplication that forces this question to have a positive answer. Let's see how to do this.

3.41 definition: *matrix multiplication*

Suppose A is an m -by- n matrix and B is an n -by- p matrix. Then AB is defined to be the m -by- p matrix whose entry in row j , column k , is given by the equation

$$(AB)_{j,k} = \sum_{r=1}^n A_{j,r}B_{r,k}.$$

Thus the entry in row j , column k , of AB is computed by taking row j of A and column k of B , multiplying together corresponding entries, and then summing.

3.43 *matrix of product of linear maps*

If $T \in \mathcal{L}(U, V)$ and $S \in \mathcal{L}(V, W)$, then $\mathcal{M}(ST) = \mathcal{M}(S)\mathcal{M}(T)$.