3A *Vector Space of Linear Maps*

3.1 definition: *linear map*

A *linear map* from V to W is a function $T\colon V \to W$ with the following properties.

additivity

$T(u+v) = Tu + Tv$ for all $u, v \in V$.

homogeneity

$T(\lambda v) = \lambda(Tv)$ for all $\lambda \in \mathbf{F}$ and all $v \in V$.

3.2 notation: $\mathcal{L}(V,W)$, $\mathcal{L}(V)$

- The set of linear maps from V to W is denoted by $\mathcal{L}(V,W)$.
- The set of linear maps from V to V is denoted by $\mathcal{L}(V)$. In other words, $\mathcal{L}(V) = \mathcal{L}(V,V)$.

3.4 *linear map lemma*

Suppose $v_1, \ldots, v_n$ is a basis of V and $w_1, \ldots, w_n \in W$. Then there exists a unique linear map $T \colon V \to W$ such that

$$Tv_k = w_k$$

for each $k = 1, \ldots, n$.

Algebraic Operations on $\mathcal{L}(V, W)$

3.5 definition: *addition and scalar multiplication on $\mathcal{L}(V, W)$*

Suppose $S, T \in \mathcal{L}(V, W)$ and $\lambda \in \mathbf{F}$. The *sum* $S + T$ and the *product* λT are the linear maps from V to W defined by

$$(S + T)(v) = Sv + Tv \quad \text{and} \quad (\lambda T)(v) = \lambda(Tv)$$

for all $v \in V$.

3.6 *$\mathcal{L}(V, W)$ is a vector space*

With the operations of addition and scalar multiplication as defined above, $\mathcal{L}(V, W)$ is a vector space.

3.7 definition: *product of linear maps*

If $T \in \mathcal{L}(U, V)$ and $S \in \mathcal{L}(V, W)$, then the *product* $ST \in \mathcal{L}(U, W)$ is defined by

$$(ST)(u) = S(Tu)$$

for all $u \in U$.

3.8 *algebraic properties of products of linear maps*

associativity

$(T_1 T_2)T_3 = T_1(T_2 T_3)$ whenever T_1, T_2, and T_3 are linear maps such that the products make sense (meaning T_3 maps into the domain of T_2, and T_2 maps into the domain of T_1).

identity

$TI = IT = T$ whenever $T \in \mathcal{L}(V, W)$; here the first I is the identity operator on V, and the second I is the identity operator on W.

distributive properties

$(S_1 + S_2)T = S_1 T + S_2 T$ and $S(T_1 + T_2) = ST_1 + ST_2$ whenever $T, T_1, T_2 \in \mathcal{L}(U, V)$ and $S, S_1, S_2 \in \mathcal{L}(V, W)$.

3.10 *linear maps take* 0 *to* 0

Suppose T is a linear map from V to W. Then $T(0) = 0$.