1A $\mathbf{R}^n$ *and* $\mathbf{C}^n$

1.1 definition: *complex numbers,* $\mathbf{C}$

- A *complex number* is an ordered pair (a, b), where $a, b \in \mathbf{R}$, but we will write this as $a + bi$.
- The set of all complex numbers is denoted by $\mathbf{C}$:

$$\mathbf{C} = \{a + bi : a, b \in \mathbf{R}\}.$$

- *Addition* and *multiplication* on $\mathbf{C}$ are defined by

$$(a + bi) + (c + di) = (a + c) + (b + d)i,$$
$$(a + bi)(c + di) = (ac - bd) + (ad + bc)i;$$

  here $a, b, c, d \in \mathbf{R}$.

1.3 *properties of complex arithmetic*

commutativity
$\alpha + \beta = \beta + \alpha$ and $\alpha\beta = \beta\alpha$ for all $\alpha, \beta \in \mathbf{C}$.

associativity
$(\alpha + \beta) + \lambda = \alpha + (\beta + \lambda)$ and $(\alpha\beta)\lambda = \alpha(\beta\lambda)$ for all $\alpha, \beta, \lambda \in \mathbf{C}$.

identities
$\lambda + 0 = \lambda$ and $\lambda 1 = \lambda$ for all $\lambda \in \mathbf{C}$.

additive inverse
For every $\alpha \in \mathbf{C}$, there exists a unique $\beta \in \mathbf{C}$ such that $\alpha + \beta = 0$.

multiplicative inverse
For every $\alpha \in \mathbf{C}$ with $\alpha \neq 0$, there exists a unique $\beta \in \mathbf{C}$ such that $\alpha\beta = 1$.

distributive property
$\lambda(\alpha + \beta) = \lambda\alpha + \lambda\beta$ for all $\lambda, \alpha, \beta \in \mathbf{C}$.

1.5 definition: $-\alpha$*, subtraction,* $1/\alpha$*, division*

Suppose $\alpha, \beta \in \mathbf{C}$.

- Let $-\alpha$ denote the additive inverse of α. Thus $-\alpha$ is the unique complex number such that

$$\alpha + (-\alpha) = 0.$$

- *Subtraction* on $\mathbf{C}$ is defined by

$$\beta - \alpha = \beta + (-\alpha).$$

- For $\alpha \neq 0$, let $1/\alpha$ and $\frac{1}{\alpha}$ denote the multiplicative inverse of α. Thus $1/\alpha$ is the unique complex number such that

$$\alpha(1/\alpha) = 1.$$

- For $\alpha \neq 0$, *division* by α is defined by

$$\beta/\alpha = \beta(1/\alpha).$$

For $\alpha \in \mathbf{F}$ and m a positive integer, we define α^m to denote the product of α with itself m times:

$$\alpha^m = \underbrace{\alpha \cdots \alpha}_{m \text{ times}}.$$

Definition

Definition 1 (Field)
*A **field** is a set $\mathbb{F}$ equipped with two binary operations (usually written as addition and multiplication) such that:*

1. *Addition and multiplication are associative: $\forall a, b, c \in \mathbb{F}$, we have*
$$a + (b + c) = (a + b) + c$$
$$(ab)c = a(bc)$$
2. *Addition and multiplication commute: $\forall a, b, c \in \mathbb{F}$, we have*
$$a + b = b + a$$
$$ab = ba$$
3. *There exist distinct additive and multiplicative identities in $\mathbb{F}$:*
$$\exists 1 \in \mathbb{F} \text{ such that } 1a = a \quad \forall a \in \mathbb{F}$$
$$\exists 0 \in \mathbb{F} \text{ such that } a + 0 = a \quad \forall a \in \mathbb{F}$$
The distinctness requirement here means that $0 \neq 1$.
4. *Additive and multiplicative inverses exist in $\mathbb{F}$:*
$$\forall a \in \mathbb{F}, \exists(-a) \in \mathbb{F} \text{ such that } a + (-a) = 0$$
$$\forall a \neq 0 \in \mathbb{F}, \exists a^{-1} \in \mathbb{F} \text{ such that } aa^{-1} = 1$$
5. *Multiplication is distributive over addition:*
$$a(b + c) = ab + ac$$

Definition

Definition 2 ($\mathbb{F}^n$)
For n a positive integer, we define:
$$\mathbb{F}^n = \{(x_1, \ldots, x_n) : x_i \in \mathbb{F}\}$$
Where the addition is component-wise.

Example

Example 3
Here is an example of addition in $\mathbb{F}^n$:
$$(x_1, \ldots, x_n) + (y_1, \ldots, y_n) = (x_1 + y_1, \ldots, x_n + y_n)$$

Notation

Notation 4
In $\mathbb{F}^n$, we let 0 denote the element $(0, 0, \ldots, 0) \in \mathbb{F}^n$, and for $x \in \mathbb{F}^n$, we let $(-x)$ denote the additive inverse of x–i.e., the element $(-x) \in \mathbb{F}^n$ such that $x + (-x) = 0$.

Definition

Definition 5 (Scaling)
For $\lambda \in \mathbb{F}$, define λx for $x \in \mathbb{F}^n$ to be:
$$\lambda x = (\lambda x_1, \ldots, \lambda x_n) \in \mathbb{F}^n$$

1B Definition of Vector Space

1.19 definition: *addition, scalar multiplication*

- An *addition* on a set V is a function that assigns an element $u + v \in V$ to each pair of elements $u, v \in V$.
- A *scalar multiplication* on a set V is a function that assigns an element $\lambda v \in V$ to each $\lambda \in \mathbf{F}$ and each $v \in V$.

1.20 definition: *vector space*

A *vector space* is a set V along with an addition on V and a scalar multiplication on V such that the following properties hold.

commutativity
$u + v = v + u$ for all $u, v \in V$.

associativity
$(u + v) + w = u + (v + w)$ and $(ab)v = a(bv)$ for all $u, v, w \in V$ and for all $a, b \in \mathbf{F}$.

additive identity
There exists an element $0 \in V$ such that $v + 0 = v$ for all $v \in V$.

additive inverse
For every $v \in V$, there exists $w \in V$ such that $v + w = 0$.

multiplicative identity
$1v = v$ for all $v \in V$.

distributive properties
$a(u + v) = au + av$ and $(a + b)v = av + bv$ for all $a, b \in \mathbf{F}$ and all $u, v \in V$.

1.21 definition: *vector, point*

Elements of a vector space are called *vectors* or *points*.

Example

Example 7
$\mathbb{F}^n$ is a vector space for any field $\mathbb{F}$. If you take F^∞ to be all sequences of elements in $\mathbb{F}$, then so is F^∞.

Notation

Notation 8
For a set S and field $\mathbb{F}$, $\mathbb{F}^S$ denotes the set of all functions from S to $\mathbb{F}$.

Example

Example 9
If we define $(f + g)(x) = f(x) + g(x)$ and $(\lambda f)(x) = \lambda(f(x))$ for all $f, g \in \mathbb{F}^S$ and $\lambda \in \mathbb{F}$, then $\mathbb{F}^S$ is a vector space.

1.26 *unique additive identity*

A vector space has a unique additive identity.

1.27 *unique additive inverse*

Every element in a vector space has a unique additive inverse.

1.28 notation: $-v$, $w - v$

Let $v, w \in V$. Then

- $-v$ denotes the additive inverse of v;
- $w - v$ is defined to be $w + (-v)$.

1.30 *the number* 0 *times a vector*

$0v = 0$ for every $v \in V$.

1.31 *a number times the vector* 0

$a0 = 0$ for every $a \in \mathbf{F}$.

1.32 *the number -1 times a vector*

$(-1)v = -v$ for every $v \in V$.

1C Subspaces

1.33 definition: *subspace*

A subset U of V is called a *subspace* of V if U is also a vector space with the same additive identity, addition, and scalar multiplication as on V.

1.34 *conditions for a subspace*

A subset U of V is a subspace of V if and only if U satisfies the following three conditions.

additive identity
$0 \in U$.

closed under addition
$u, w \in U$ implies $u + w \in U$.

closed under scalar multiplication
$a \in \mathbf{F}$ and $u \in U$ implies $au \in U$.

1.36 definition: *sum of subspaces*

Suppose $V_1, \ldots, V_m$ are subspaces of V. The *sum* of $V_1, \ldots, V_m$, denoted by $V_1 + \cdots + V_m$, is the set of all possible sums of elements of $V_1, \ldots, V_m$. More precisely,

$$V_1 + \cdots + V_m = \{v_1 + \cdots + v_m : v_1 \in V_1, \ldots, v_m \in V_m\}.$$

1.40 *sum of subspaces is the smallest containing subspace*

Suppose $V_1, \dots, V_m$ are subspaces of V. Then $V_1 + \cdots + V_m$ is the smallest subspace of V containing $V_1, \dots, V_m$.

1.41 definition: *direct sum,* $\oplus$

Suppose $V_1, \ldots, V_m$ are subspaces of V.

- The sum $V_1 + \cdots + V_m$ is called a *direct sum* if each element of $V_1 + \cdots + V_m$ can be written in only one way as a sum $v_1 + \cdots + v_m$, where each $v_k \in V_k$.
- If $V_1 + \cdots + V_m$ is a direct sum, then $V_1 \oplus \cdots \oplus V_m$ denotes $V_1 + \cdots + V_m$, with the $\oplus$ notation serving as an indication that this is a direct sum.

1.45 *condition for a direct sum*

Suppose $V_1, \ldots, V_m$ are subspaces of V. Then $V_1 + \cdots + V_m$ is a direct sum if and only if the only way to write 0 as a sum $v_1 + \cdots + v_m$, where each $v_k \in V_k$, is by taking each v_k equal to 0.

1.46 *direct sum of two subspaces*

Suppose U and W are subspaces of V. Then

$$U + W \text{ is a direct sum} \iff U \cap W = \{0\}.$$

2A *Span and Linear Independence*

2.2 definition: *linear combination*

A *linear combination* of a list $v_1, \ldots, v_m$ of vectors in V is a vector of the form

$$a_1 v_1 + \cdots + a_m v_m,$$

where $a_1, \ldots, a_m \in \mathbf{F}$.

2.4 definition: *span*

The set of all linear combinations of a list of vectors $v_1, \ldots, v_m$ in V is called the *span* of $v_1, \ldots, v_m$, denoted by $\operatorname{span}(v_1, \ldots, v_m)$. In other words,

$$\operatorname{span}(v_1, \ldots, v_m) = \{a_1 v_1 + \cdots + a_m v_m : a_1, \ldots, a_m \in \mathbf{F}\}.$$

The span of the empty list $(\,)$ is defined to be $\{0\}$.

2.6 *span is the smallest containing subspace*

The span of a list of vectors in V is the smallest subspace of V containing all vectors in the list.

2.7 definition: *spans*

If $\operatorname{span}(v_1, \ldots, v_m)$ equals V, we say that the list $v_1, \ldots, v_m$ *spans* V.

2.9 definition: *finite-dimensional vector space*

A vector space is called *finite-dimensional* if some list of vectors in it spans the space.

Recall that by definition every list has finite length.

2.13 definition: *infinite-dimensional vector space*

A vector space is called *infinite-dimensional* if it is not finite-dimensional.

2.10 definition: *polynomial,* $\mathcal{P}(\mathbf{F})$

- A function $p\colon \mathbf{F} \to \mathbf{F}$ is called a *polynomial* with coefficients in $\mathbf{F}$ if there exist $a_0, \ldots, a_m \in \mathbf{F}$ such that

$$p(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_m z^m$$

  for all $z \in \mathbf{F}$.
- $\mathcal{P}(\mathbf{F})$ is the set of all polynomials with coefficients in $\mathbf{F}$.

2.12 notation: $\mathcal{P}_m(\mathbf{F})$

For m a nonnegative integer, $\mathcal{P}_m(\mathbf{F})$ denotes the set of all polynomials with coefficients in $\mathbf{F}$ and degree at most m.

2.15 definition: *linearly independent*

- A list $v_1, \ldots, v_m$ of vectors in V is called *linearly independent* if the only choice of $a_1, \ldots, a_m \in \mathbf{F}$ that makes

$$a_1 v_1 + \cdots + a_m v_m = 0$$

  is $a_1 = \cdots = a_m = 0$.
- The empty list $(\,)$ is also declared to be linearly independent.

2.17 definition: *linearly dependent*

- A list of vectors in V is called *linearly dependent* if it is not linearly independent.
- In other words, a list $v_1, \ldots, v_m$ of vectors in V is linearly dependent if there exist $a_1, \ldots, a_m \in \mathbf{F}$, not all 0, such that $a_1 v_1 + \cdots + a_m v_m = 0$.

2.19 *linear dependence lemma*

Suppose $v_1, \ldots, v_m$ is a linearly dependent list in V. Then there exists $k \in \{1, 2, \ldots, m\}$ such that

$$v_k \in \operatorname{span}(v_1, \ldots, v_{k-1}).$$

Furthermore, if k satisfies the condition above and the k^{th} term is removed from $v_1, \ldots, v_m$, then the span of the remaining list equals $\operatorname{span}(v_1, \ldots, v_m)$.

2.22 *length of linearly independent list $\leq$ length of spanning list*

In a finite-dimensional vector space, the length of every linearly independent list of vectors is less than or equal to the length of every spanning list of vectors.

2.25 *finite-dimensional subspaces*

Every subspace of a finite-dimensional vector space is finite-dimensional.

2B Bases

2.26 definition: *basis*

A *basis* of V is a list of vectors in V that is linearly independent and spans V.

2.28 *criterion for basis*

A list $v_1, \ldots, v_n$ of vectors in V is a basis of V if and only if every $v \in V$ can be written uniquely in the form

2.29
$$v = a_1 v_1 + \cdots + a_n v_n,$$

where $a_1, \ldots, a_n \in \mathbf{F}$.

2.30 *every spanning list contains a basis*

Every spanning list in a vector space can be reduced to a basis of the vector space.

2.31 *basis of finite-dimensional vector space*

Every finite-dimensional vector space has a basis.

2.32 *every linearly independent list extends to a basis*

Every linearly independent list of vectors in a finite-dimensional vector space can be extended to a basis of the vector space.

2.33 *every subspace of V is part of a direct sum equal to V*

Suppose V is finite-dimensional and U is a subspace of V. Then there is a subspace W of V such that $V = U \oplus W$.

2C Dimension

2.22 *length of linearly independent list $\leq$ length of spanning list*

In a finite-dimensional vector space, the length of every linearly independent list of vectors is less than or equal to the length of every spanning list of vectors.

2.34 *basis length does not depend on basis*

Any two bases of a finite-dimensional vector space have the same length.

2.35 definition: *dimension*, $\dim V$

- The *dimension* of a finite-dimensional vector space is the length of any basis of the vector space.
- The dimension of a finite-dimensional vector space V is denoted by $\dim V$.

2.37 *dimension of a subspace*

If V is finite-dimensional and U is a subspace of V, then $\dim U \leq \dim V$.

2.38 *linearly independent list of the right length is a basis*

Suppose V is finite-dimensional. Then every linearly independent list of vectors in V of length $\dim V$ is a basis of V.

2.39 *subspace of full dimension equals the whole space*

Suppose that V is finite-dimensional and U is a subspace of V such that $\dim U = \dim V$. Then $U = V$.

2.42 *spanning list of the right length is a basis*

Suppose V is finite-dimensional. Then every spanning list of vectors in V of length $\dim V$ is a basis of V.

2.43 *dimension of a sum*

If V_1 and V_2 are subspaces of a finite-dimensional vector space, then

$$\dim(V_1 + V_2) = \dim V_1 + \dim V_2 - \dim(V_1 \cap V_2).$$

sets	**vector spaces**
S is a finite set	V is a finite-dimensional vector space
$\#S$	$\dim V$
for subsets S_1, S_2 of S, the union $S_1 \cup S_2$ is the smallest subset of S containing S_1 and S_2	for subspaces V_1, V_2 of V, the sum $V_1 + V_2$ is the smallest subspace of V containing V_1 and V_2
$\#(S_1 \cup S_2)$ $= \#S_1 + \#S_2 - \#(S_1 \cap S_2)$	$\dim(V_1 + V_2)$ $= \dim V_1 + \dim V_2 - \dim(V_1 \cap V_2)$
$\#(S_1 \cup S_2) = \#S_1 + \#S_2$ $\iff S_1 \cap S_2 = \emptyset$	$\dim(V_1 + V_2) = \dim V_1 + \dim V_2$ $\iff V_1 \cap V_2 = \{0\}$
$S_1 \cup \cdots \cup S_m$ is a disjoint union $\iff$ $\#(S_1 \cup \cdots \cup S_m) = \#S_1 + \cdots + \#S_m$	$V_1 + \cdots + V_m$ is a direct sum $\iff$ $\dim(V_1 + \cdots + V_m)$ $= \dim V_1 + \cdots + \dim V_m$

3A Vector Space of Linear Maps

3.1 definition: *linear map*

A *linear map* from V to W is a function $T\colon V \to W$ with the following properties.

additivity
$T(u+v) = Tu + Tv$ for all $u, v \in V$.

homogeneity
$T(\lambda v) = \lambda(Tv)$ for all $\lambda \in \mathbf{F}$ and all $v \in V$.

3.2 notation: $\mathcal{L}(V,W)$, $\mathcal{L}(V)$

- The set of linear maps from V to W is denoted by $\mathcal{L}(V,W)$.
- The set of linear maps from V to V is denoted by $\mathcal{L}(V)$. In other words, $\mathcal{L}(V) = \mathcal{L}(V,V)$.

3.4 *linear map lemma*

Suppose $v_1, \ldots, v_n$ is a basis of V and $w_1, \ldots, w_n \in W$. Then there exists a unique linear map $T\colon V \to W$ such that

$$Tv_k = w_k$$

for each $k = 1, \ldots, n$.

Algebraic Operations on $\mathcal{L}(V, W)$

3.5 definition: *addition and scalar multiplication on $\mathcal{L}(V, W)$*

Suppose $S, T \in \mathcal{L}(V, W)$ and $\lambda \in \mathbf{F}$. The *sum* $S + T$ and the *product* λT are the linear maps from V to W defined by

$$(S + T)(v) = Sv + Tv \quad \text{and} \quad (\lambda T)(v) = \lambda(Tv)$$

for all $v \in V$.

3.6 *$\mathcal{L}(V, W)$ is a vector space*

With the operations of addition and scalar multiplication as defined above, $\mathcal{L}(V, W)$ is a vector space.

3.7 definition: *product of linear maps*

If $T \in \mathcal{L}(U, V)$ and $S \in \mathcal{L}(V, W)$, then the *product* $ST \in \mathcal{L}(U, W)$ is defined by

$$(ST)(u) = S(Tu)$$

for all $u \in U$.

3.8 *algebraic properties of products of linear maps*

associativity

$(T_1T_2)T_3 = T_1(T_2T_3)$ whenever T_1, T_2, and T_3 are linear maps such that the products make sense (meaning T_3 maps into the domain of T_2, and T_2 maps into the domain of T_1).

identity

$TI = IT = T$ whenever $T \in \mathcal{L}(V, W)$; here the first I is the identity operator on V, and the second I is the identity operator on W.

distributive properties

$(S_1 + S_2)T = S_1T + S_2T$ and $S(T_1 + T_2) = ST_1 + ST_2$ whenever $T, T_1, T_2 \in \mathcal{L}(U, V)$ and $S, S_1, S_2 \in \mathcal{L}(V, W)$.

3.10 *linear maps take* 0 *to* 0

Suppose T is a linear map from V to W. Then $T(0) = 0$.

3B Null Spaces and Ranges

Null Space and Injectivity

3.11 definition: *null space,* null T

For $T \in \mathcal{L}(V, W)$, the *null space* of T, denoted by null T, is the subset of V consisting of those vectors that T maps to 0:

$$\operatorname{null} T = \{v \in V : Tv = 0\}.$$

3.13 *the null space is a subspace*

Suppose $T \in \mathcal{L}(V, W)$. Then null T is a subspace of V.

3.14 definition: *injective*

A function $T: V \to W$ is called *injective* if $Tu = Tv$ implies $u = v$.

3.15 *injectivity* $\Longleftrightarrow$ *null space equals* $\{0\}$

Let $T \in \mathcal{L}(V, W)$. Then T is injective if and only if null $T = \{0\}$.

Range and Surjectivity

3.16 definition: *range*

For $T \in \mathcal{L}(V, W)$, the *range* of T is the subset of W consisting of those vectors that are equal to Tv for some $v \in V$:

$$\operatorname{range} T = \{Tv : v \in V\}.$$

3.18 *the range is a subspace*

If $T \in \mathcal{L}(V, W)$, then $\operatorname{range} T$ is a subspace of W.

3.19 definition: *surjective*

A function $T \colon V \to W$ is called *surjective* if its range equals W.

3.20 example: *surjectivity depends on the target space*

The differentiation map $D \in \mathcal{L}(\mathcal{P}_5(\mathbf{R}))$ defined by $Dp = p'$ is not surjective, because the polynomial x^5 is not in the range of D. However, the differentiation map $S \in \mathcal{L}(\mathcal{P}_5(\mathbf{R}), \mathcal{P}_4(\mathbf{R}))$ defined by $Sp = p'$ is surjective, because its range equals $\mathcal{P}_4(\mathbf{R})$, which is the vector space into which S maps.

Fundamental Theorem of Linear Maps

3.21 *fundamental theorem of linear maps*

Suppose V is finite-dimensional and $T \in \mathcal{L}(V, W)$. Then range T is finite-dimensional and

$$\dim V = \dim \operatorname{null} T + \dim \operatorname{range} T.$$

3.22 *linear map to a lower-dimensional space is not injective*

Suppose V and W are finite-dimensional vector spaces such that $\dim V > \dim W$. Then no linear map from V to W is injective.

3.24 *linear map to a higher-dimensional space is not surjective*

Suppose V and W are finite-dimensional vector spaces such that $\dim V < \dim W$. Then no linear map from V to W is surjective.

3.26 *homogeneous system of linear equations*

A homogeneous system of linear equations with more variables than equations has nonzero solutions.

3.28 *inhomogeneous system of linear equations*

An inhomogeneous system of linear equations with more equations than variables has no solution for some choice of the constant terms.

3C Matrices

Representing a Linear Map by a Matrix

3.29 definition: *matrix, $A_{j,k}$*

Suppose m and n are nonnegative integers. An m-by-n *matrix* A is a rectangular array of elements of $\mathbf{F}$ with m rows and n columns:

$$A = \begin{pmatrix} A_{1,1} & \cdots & A_{1,n} \\ \vdots & & \vdots \\ A_{m,1} & \cdots & A_{m,n} \end{pmatrix}.$$

The notation $A_{j,k}$ denotes the entry in row j, column k of A.

3.31 definition: *matrix of a linear map, $\mathcal{M}(T)$*

Suppose $T \in \mathcal{L}(V, W)$ and $v_1, \dots, v_n$ is a basis of V and $w_1, \dots, w_m$ is a basis of W. The *matrix of* T with respect to these bases is the m-by-n matrix $\mathcal{M}(T)$ whose entries $A_{j,k}$ are defined by

$$Tv_k = A_{1,k}w_1 + \cdots + A_{m,k}w_m.$$

If the bases $v_1, \dots, v_n$ and $w_1, \dots, w_m$ are not clear from the context, then the notation $\mathcal{M}(T, (v_1, \dots, v_n), (w_1, \dots, w_m))$ is used.

Idea

$$\mathcal{M}(T) = \begin{array}{c} \\ w_1 \\ \vdots \\ w_m \end{array} \begin{array}{c} \begin{array}{ccccc} v_1 & \cdots & v_k & \cdots & v_n \end{array} \\ \begin{pmatrix} & & A_{1,k} & & \\ & & \vdots & & \\ & & A_{m,k} & & \end{pmatrix} \end{array}.$$

The k^{th} column of $\mathcal{M}(T)$ consists of the scalars needed to write Tv_k as a linear combination of $w_1, \dots, w_m$:

$$Tv_k = \sum_{j=1}^{m} A_{j,k}w_j.$$

Ex

Suppose $T \in \mathcal{L}(\mathbf{F}^2, \mathbf{F}^3)$ is defined by

$$T(x, y) = (x + 3y, 2x + 5y, 7x + 9y).$$

Ex

Suppose $D \in \mathcal{L}(\mathcal{P}_3(\mathbf{R}), \mathcal{P}_2(\mathbf{R}))$ is the differentiation map defined by $Dp = p'$.

Addition and Scalar Multiplication of Matrices

3.34 definition: *matrix addition*

The *sum of two matrices of the same size* is the matrix obtained by adding corresponding entries in the matrices:

$$\begin{pmatrix} A_{1,1} & \cdots & A_{1,n} \\ \vdots & & \vdots \\ A_{m,1} & \cdots & A_{m,n} \end{pmatrix} + \begin{pmatrix} C_{1,1} & \cdots & C_{1,n} \\ \vdots & & \vdots \\ C_{m,1} & \cdots & C_{m,n} \end{pmatrix}$$
$$= \begin{pmatrix} A_{1,1} + C_{1,1} & \cdots & A_{1,n} + C_{1,n} \\ \vdots & & \vdots \\ A_{m,1} + C_{m,1} & \cdots & A_{m,n} + C_{m,n} \end{pmatrix}.$$

3.35 *matrix of the sum of linear maps*

Suppose $S, T \in \mathcal{L}(V, W)$. Then $\mathcal{M}(S + T) = \mathcal{M}(S) + \mathcal{M}(T)$.

3.36 definition: *scalar multiplication of a matrix*

The product of a scalar and a matrix is the matrix obtained by multiplying each entry in the matrix by the scalar:

$$\lambda \begin{pmatrix} A_{1,1} & \cdots & A_{1,n} \\ \vdots & & \vdots \\ A_{m,1} & \cdots & A_{m,n} \end{pmatrix} = \begin{pmatrix} \lambda A_{1,1} & \cdots & \lambda A_{1,n} \\ \vdots & & \vdots \\ \lambda A_{m,1} & \cdots & \lambda A_{m,n} \end{pmatrix}.$$

3.38 *the matrix of a scalar times a linear map*

Suppose $\lambda \in \mathbf{F}$ and $T \in \mathcal{L}(V, W)$. Then $\mathcal{M}(\lambda T) = \lambda \mathcal{M}(T)$.

3.39 notation: $\mathbf{F}^{m,n}$

For m and n positive integers, the set of all m-by-n matrices with entries in $\mathbf{F}$ is denoted by $\mathbf{F}^{m,n}$.

3.40 $\dim \mathbf{F}^{m,n} = mn$

Suppose m and n are positive integers. With addition and scalar multiplication defined as above, $\mathbf{F}^{m,n}$ is a vector space of dimension mn.

Matrix Multiplication

Consider linear maps $T: U \to V$ and $S: V \to W$. The composition ST is a linear map from U to W. Does $\mathcal{M}(ST)$ equal $\mathcal{M}(S)\mathcal{M}(T)$? This question does not yet make sense because we have not defined the product of two matrices. We will choose a definition of matrix multiplication that forces this question to have a positive answer. Let's see how to do this.

3.41 definition: *matrix multiplication*

Suppose A is an m-by-n matrix and B is an n-by-p matrix. Then AB is defined to be the m-by-p matrix whose entry in row j, column k, is given by the equation

$$(AB)_{j,k} = \sum_{r=1}^{n} A_{j,r} B_{r,k}.$$

Thus the entry in row j, column k, of AB is computed by taking row j of A and column k of B, multiplying together corresponding entries, and then summing.

3.43 *matrix of product of linear maps*

If $T \in \mathcal{L}(U, V)$ and $S \in \mathcal{L}(V, W)$, then $\mathcal{M}(ST) = \mathcal{M}(S)\mathcal{M}(T)$.

3.44 notation: $A_{j,\cdot}$, $A_{\cdot,k}$

Suppose A is an m-by-n matrix.

- If $1 \leq j \leq m$, then $A_{j,\cdot}$ denotes the 1-by-n matrix consisting of row j of A.
- If $1 \leq k \leq n$, then $A_{\cdot,k}$ denotes the m-by-1 matrix consisting of column k of A.

3.46 ***entry of matrix product equals row times column***

Suppose A is an m-by-n matrix and B is an n-by-p matrix. Then

$$(AB)_{j,k} = A_{j,\cdot} B_{\cdot,k}$$

if $1 \leq j \leq m$ and $1 \leq k \leq p$. In other words, the entry in row j, column k, of AB equals (row j of A) times (column k of B).

3.48 ***column of matrix product equals matrix times column***

Suppose A is an m-by-n matrix and B is an n-by-p matrix. Then

$$(AB)_{\cdot,k} = AB_{\cdot,k}$$

if $1 \leq k \leq p$. In other words, column k of AB equals A times column k of B.

3.50 *linear combination of columns*

Suppose A is an m-by-n matrix and $b = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$ is an n-by-1 matrix. Then

$$Ab = b_1 A_{\cdot,1} + \cdots + b_n A_{\cdot,n}.$$

In other words, Ab is a linear combination of the columns of A, with the scalars that multiply the columns coming from b.

3.51 *matrix multiplication as linear combinations of columns*

Suppose C is an m-by-c matrix and R is a c-by-n matrix.

(a) If $k \in \{1, \dots, n\}$, then column k of CR is a linear combination of the columns of C, with the coefficients of this linear combination coming from column k of R.

(b) If $j \in \{1, \dots, m\}$, then row j of CR is a linear combination of the rows of R, with the coefficients of this linear combination coming from row j of C.

3.52 definition: *column rank, row rank*

Suppose A is an m-by-n matrix with entries in $\mathbf{F}$.

- The *column rank* of A is the dimension of the span of the columns of A in $\mathbf{F}^{m,1}$.
- The *row rank* of A is the dimension of the span of the rows of A in $\mathbf{F}^{1,n}$.

$\boxed{\text{Ex}}$

Suppose

$$A = \begin{pmatrix} 4 & 7 & 1 & 8 \\ 3 & 5 & 2 & 9 \end{pmatrix}.$$

The column rank of A is the dimension of

$$\operatorname{span}\left(\begin{pmatrix} 4 \\ 3 \end{pmatrix}, \begin{pmatrix} 7 \\ 5 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 8 \\ 9 \end{pmatrix}\right)$$

The row rank of A is the dimension of

$$\operatorname{span}\big(\begin{pmatrix} 4 & 7 & 1 & 8 \end{pmatrix}, \begin{pmatrix} 3 & 5 & 2 & 9 \end{pmatrix}\big)$$

3.54 definition: *transpose, A^{t}*

The *transpose* of a matrix A, denoted by A^{t}, is the matrix obtained from A by interchanging rows and columns. Specifically, if A is an m-by-n matrix, then A^{t} is the n-by-m matrix whose entries are given by the equation

$$(A^{\mathrm{t}})_{k,j} = A_{j,k}.$$

3.56 *column–row factorization*

Suppose A is an m-by-n matrix with entries in $\mathbf{F}$ and column rank $c \geq 1$. Then there exist an m-by-c matrix C and a c-by-n matrix R, both with entries in $\mathbf{F}$, such that $A = CR$.

3.57 *column rank equals row rank*

Suppose $A \in \mathbf{F}^{m,n}$. Then the column rank of A equals the row rank of A.

3.58 definition: *rank*

The *rank* of a matrix $A \in \mathbf{F}^{m,n}$ is the column rank of A.

3D Invertibility and Isomorphisms

Invertible Linear Maps

3.59 definition: *invertible, inverse*

- A linear map $T \in \mathcal{L}(V, W)$ is called *invertible* if there exists a linear map $S \in \mathcal{L}(W, V)$ such that ST equals the identity operator on V and TS equals the identity operator on W.
- A linear map $S \in \mathcal{L}(W, V)$ satisfying $ST = I$ and $TS = I$ is called an *inverse* of T (note that the first I is the identity operator on V and the second I is the identity operator on W).

3.60 *inverse is unique*

An invertible linear map has a unique inverse.

3.61 notation: T^{-1}

If T is invertible, then its inverse is denoted by T^{-1}. In other words, if $T \in \mathcal{L}(V, W)$ is invertible, then T^{-1} is the unique element of $\mathcal{L}(W, V)$ such that $T^{-1}T = I$ and $TT^{-1} = I$.

3.63 *invertibility* $\iff$ *injectivity and surjectivity*

A linear map is invertible if and only if it is injective and surjective.

3.65 *injectivity is equivalent to surjectivity (if* $\dim V = \dim W < \infty$)

Suppose that V and W are finite-dimensional vector spaces, $\dim V = \dim W$, and $T \in \mathcal{L}(V, W)$. Then

$$T \text{ is invertible} \iff T \text{ is injective} \iff T \text{ is surjective.}$$

$\boxed{\text{Ex}}$ Define $T \in \mathcal{L}(\mathcal{P}_m(\mathbb{R}))$ by $(Tp)(x) = ((x^2+5x+7)p(x))''$.

3.68 $ST = I \iff TS = I$ (*on vector spaces of the same dimension*)

Suppose V and W are finite-dimensional vector spaces of the same dimension, $S \in \mathcal{L}(V, W)$, and $T \in \mathcal{L}(W, V)$. Then $ST = I$ if and only if $TS = I$.

Isomorphic Vector Spaces

3.69 definition: *isomorphism, isomorphic*

- An *isomorphism* is an invertible linear map.
- Two vector spaces are called *isomorphic* if there is an isomorphism from one vector space onto the other one.

3.70 ***dimension shows whether vector spaces are isomorphic***

Two finite-dimensional vector spaces over $\mathbf{F}$ are isomorphic if and only if they have the same dimension.

3.71 $\mathcal{L}(V, W)$ *and* $\mathbf{F}^{m,n}$ *are isomorphic*

Suppose $v_1, \ldots, v_n$ is a basis of V and $w_1, \ldots, w_m$ is a basis of W. Then $\mathcal{M}$ is an isomorphism between $\mathcal{L}(V, W)$ and $\mathbf{F}^{m,n}$.

3.72 $\dim \mathcal{L}(V, W) = (\dim V)(\dim W)$

Suppose V and W are finite-dimensional. Then $\mathcal{L}(V, W)$ is finite-dimensional and

$$\dim \mathcal{L}(V, W) = (\dim V)(\dim W).$$

Linear Maps Thought of as Matrix Multiplication

3.73 definition: *matrix of a vector, $\mathcal{M}(v)$*

Suppose $v \in V$ and $v_1, \ldots, v_n$ is a basis of V. The *matrix of v* with respect to this basis is the n-by-1 matrix

$$\mathcal{M}(v) = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix},$$

where $b_1, \ldots, b_n$ are the scalars such that

$$v = b_1 v_1 + \cdots + b_n v_n.$$

3.75 $\mathcal{M}(T)_{\cdot,k} = \mathcal{M}(Tv_k)$.

Suppose $T \in \mathcal{L}(V, W)$ and $v_1, \ldots, v_n$ is a basis of V and $w_1, \ldots, w_m$ is a basis of W. Let $1 \leq k \leq n$. Then the k^{th} column of $\mathcal{M}(T)$, which is denoted by $\mathcal{M}(T)_{\cdot,k}$, equals $\mathcal{M}(Tv_k)$.

3.76 *linear maps act like matrix multiplication*

Suppose $T \in \mathcal{L}(V, W)$ and $v \in V$. Suppose $v_1, \ldots, v_n$ is a basis of V and $w_1, \ldots, w_m$ is a basis of W. Then

$$\mathcal{M}(Tv) = \mathcal{M}(T)\mathcal{M}(v).$$

3.78 *dimension of* range T *equals column rank of* $\mathcal{M}(T)$

Suppose V and W are finite-dimensional and $T \in \mathcal{L}(V, W)$. Then $\dim \operatorname{range} T$ equals the column rank of $\mathcal{M}(T)$.

Change of Basis

3.81 *matrix of product of linear maps*

Suppose $T \in \mathcal{L}(U, V)$ and $S \in \mathcal{L}(V, W)$. If $u_1, \ldots, u_m$ is a basis of U, $v_1, \ldots, v_n$ is a basis of V, and $w_1, \ldots, w_p$ is a basis of W, then

$$\mathcal{M}(ST, (u_1, \ldots, u_m), (w_1, \ldots, w_p)) = \\ \mathcal{M}(S, (v_1, \ldots, v_n), (w_1, \ldots, w_p))\mathcal{M}(T, (u_1, \ldots, u_m), (v_1, \ldots, v_n)).$$

3.82 *matrix of identity operator with respect to two bases*

Suppose that $u_1, \ldots, u_n$ and $v_1, \ldots, v_n$ are bases of V. Then the matrices

$$\mathcal{M}(I, (u_1, \ldots, u_n), (v_1, \ldots, v_n)) \quad \text{and} \quad \mathcal{M}(I, (v_1, \ldots, v_n), (u_1, \ldots, u_n))$$

are invertible, and each is the inverse of the other.

3.84 *change-of-basis formula*

Suppose $T \in \mathcal{L}(V)$. Suppose $u_1, \ldots, u_n$ and $v_1, \ldots, v_n$ are bases of V. Let

$$A = \mathcal{M}(T, (u_1, \ldots, u_n)) \quad \text{and} \quad B = \mathcal{M}(T, (v_1, \ldots, v_n))$$

and $C = \mathcal{M}(I, (u_1, \ldots, u_n), (v_1, \ldots, v_n))$. Then

$$A = C^{-1}BC.$$

3.86 *matrix of inverse equals inverse of matrix*

Suppose that $v_1, \ldots, v_n$ is a basis of V and $T \in \mathcal{L}(V)$ is invertible. Then $\mathcal{M}(T^{-1}) = (\mathcal{M}(T))^{-1}$, where both matrices are with respect to the basis $v_1, \ldots, v_n$.

3E Products and Quotients of Vector Spaces

Products of Vector Spaces

3.87 definition: *product of vector spaces*

Suppose $V_1, \dots, V_m$ are vector spaces over $\mathbf{F}$.

- The *product* $V_1 \times \cdots \times V_m$ is defined by

$$V_1 \times \cdots \times V_m = \{(v_1, \dots, v_m) : v_1 \in V_1, \dots, v_m \in V_m\}.$$

- Addition on $V_1 \times \cdots \times V_m$ is defined by

$$(u_1, \dots, u_m) + (v_1, \dots, v_m) = (u_1 + v_1, \dots, u_m + v_m).$$

- Scalar multiplication on $V_1 \times \cdots \times V_m$ is defined by

$$\lambda(v_1, \dots, v_m) = (\lambda v_1, \dots, \lambda v_m).$$

3.89 *product of vector spaces is a vector space*

Suppose $V_1, \dots, V_m$ are vector spaces over $\mathbf{F}$. Then $V_1 \times \cdots \times V_m$ is a vector space over $\mathbf{F}$.

3.92 *dimension of a product is the sum of dimensions*

Suppose $V_1, \dots, V_m$ are finite-dimensional vector spaces. Then $V_1 \times \cdots \times V_m$ is finite-dimensional and

$$\dim(V_1 \times \cdots \times V_m) = \dim V_1 + \cdots + \dim V_m.$$

3.93 *products and direct sums*

Suppose that $V_1, \ldots, V_m$ are subspaces of V. Define a linear map $\Gamma : V_1 \times \cdots \times V_m \to V_1 + \cdots + V_m$ by

$$\Gamma(v_1, \ldots, v_m) = v_1 + \cdots + v_m.$$

Then $V_1 + \cdots + V_m$ is a direct sum if and only if Γ is injective.

3.94 *a sum is a direct sum if and only if dimensions add up*

Suppose V is finite-dimensional and $V_1, \ldots, V_m$ are subspaces of V. Then $V_1 + \cdots + V_m$ is a direct sum if and only if

$$\dim(V_1 + \cdots + V_m) = \dim V_1 + \cdots + \dim V_m.$$

Quotient Spaces

3.95 notation: $v + U$

Suppose $v \in V$ and $U \subseteq V$. Then $v + U$ is the subset of V defined by

$$v + U = \{v + u : u \in U\}.$$

3.97 definition: *translate*

For $v \in V$ and U a subset of V, the set $v + U$ is said to be a *translate* of U.

3.99 definition: *quotient space*, V/U

Suppose U is a subspace of V. Then the *quotient space* V/U is the set of all translates of U. Thus

$$V/U = \{v + U : v \in V\}.$$

3.101 *two translates of a subspace are equal or disjoint*

Suppose U is a subspace of V and $v, w \in V$. Then

$$v - w \in U \iff v + U = w + U \iff (v + U) \cap (w + U) \neq \emptyset.$$

3.102 definition: *addition and scalar multiplication on V/U*

Suppose U is a subspace of V. Then *addition* and *scalar multiplication* are defined on V/U by

$$(v+U)+(w+U)=(v+w)+U$$
$$\lambda(v+U)=(\lambda v)+U$$

for all $v,w\in V$ and all $\lambda\in\mathbf{F}$.

3.103 *quotient space is a vector space*

Suppose U is a subspace of V. Then V/U, with the operations of addition and scalar multiplication as defined above, is a vector space.

3.104 definition: *quotient map*, π

Suppose U is a subspace of V. The *quotient map* $\pi\colon V \to V/U$ is the linear map defined by

$$\pi(v) = v + U$$

for each $v \in V$.

3.105 *dimension of quotient space*

Suppose V is finite-dimensional and U is a subspace of V. Then

$$\dim V/U = \dim V - \dim U.$$

3.106 notation: $\widetilde{T}$

Suppose $T \in \mathcal{L}(V, W)$. Define $\widetilde{T}\colon V/(\operatorname{null} T) \to W$ by

$$\widetilde{T}(v + \operatorname{null} T) = Tv.$$

3.107 *null space and range of $\widetilde{T}$*

Suppose $T \in \mathcal{L}(V, W)$. Then

(a) $\widetilde{T} \circ \pi = T$, where π is the quotient map of V onto $V/(\operatorname{null} T)$;

(b) $\widetilde{T}$ is injective;

(c) $\operatorname{range} \widetilde{T} = \operatorname{range} T$;

(d) $V/(\operatorname{null} T)$ and $\operatorname{range} T$ are isomorphic vector spaces.

3F Duality

Dual Space and Dual Map

3.108 definition: *linear functional*

A *linear functional* on V is a linear map from V to $\mathbf{F}$. In other words, a linear functional is an element of $\mathcal{L}(V, \mathbf{F})$.

3.110 definition: *dual space, V'*

The *dual space* of V, denoted by V', is the vector space of all linear functionals on V. In other words, $V' = \mathcal{L}(V, \mathbf{F})$.

3.111 $\dim V' = \dim V$

Suppose V is finite-dimensional. Then V' is also finite-dimensional and

$$\dim V' = \dim V.$$

3.112 definition: *dual basis*

If $v_1, \ldots, v_n$ is a basis of V, then the *dual basis* of $v_1, \ldots, v_n$ is the list $\varphi_1, \ldots, \varphi_n$ of elements of V', where each φ_j is the linear functional on V such that

$$\varphi_j(v_k) = \begin{cases} 1 & \text{if } k = j, \\ 0 & \text{if } k \neq j. \end{cases}$$

3.114 *dual basis gives coefficients for linear combination*

Suppose $v_1, ..., v_n$ is a basis of V and $\varphi_1, ..., \varphi_n$ is the dual basis. Then

$$v = \varphi_1(v)v_1 + \cdots + \varphi_n(v)v_n$$

for each $v \in V$.

3.116 *dual basis is a basis of the dual space*

Suppose V is finite-dimensional. Then the dual basis of a basis of V is a basis of V'.

3.118 definition: *dual map,* T'

Suppose $T \in \mathcal{L}(V, W)$. The *dual map* of T is the linear map $T' \in \mathcal{L}(W', V')$ defined for each $\varphi \in W'$ by

$$T'(\varphi) = \varphi \circ T.$$

3.120 *algebraic properties of dual maps*

Suppose $T \in \mathcal{L}(V, W)$. Then

(a) $(S + T)' = S' + T'$ for all $S \in \mathcal{L}(V, W)$;

(b) $(\lambda T)' = \lambda T'$ for all $\lambda \in \mathbf{F}$;

(c) $(ST)' = T'S'$ for all $S \in \mathcal{L}(W, U)$.

Null Space and Range of Dual of Linear Map

3.121 definition: *annihilator, U^0*

For $U \subseteq V$, the *annihilator* of U, denoted by U^0, is defined by

$$U^0 = \{\varphi \in V' : \varphi(u) = 0 \text{ for all } u \in U\}.$$

3.124 *the annihilator is a subspace*

Suppose $U \subseteq V$. Then U^0 is a subspace of V'.

3.125 *dimension of the annihilator*

Suppose V is finite-dimensional and U is a subspace of V. Then

$$\dim U^0 = \dim V - \dim U.$$

3.127 *condition for the annihilator to equal* $\{0\}$ *or the whole space*

Suppose V is finite-dimensional and U is a subspace of V. Then

(a) $U^0 = \{0\} \iff U = V$;

(b) $U^0 = V' \iff U = \{0\}$.

3.128 *the null space of T'*

Suppose V and W are finite-dimensional and $T \in \mathcal{L}(V, W)$. Then

(a) $\operatorname{null} T' = (\operatorname{range} T)^0$;

(b) $\dim \operatorname{null} T' = \dim \operatorname{null} T + \dim W - \dim V$.

3.129 *T surjective is equivalent to T' injective*

Suppose V and W are finite-dimensional and $T \in \mathcal{L}(V, W)$. Then

$$T \text{ is surjective } \iff T' \text{ is injective.}$$

3.130 *the range of T'*

Suppose V and W are finite-dimensional and $T \in \mathcal{L}(V, W)$. Then

(a) $\dim \operatorname{range} T' = \dim \operatorname{range} T$;

(b) $\operatorname{range} T' = (\operatorname{null} T)^0$.

3.131 *T injective is equivalent to T' surjective*

Suppose V and W are finite-dimensional and $T \in \mathcal{L}(V, W)$. Then

$$T \text{ is injective} \iff T' \text{ is surjective.}$$

Matrix of Dual of Linear Map

3.132 *matrix of T' is transpose of matrix of T*

Suppose V and W are finite-dimensional and $T \in \mathcal{L}(V, W)$. Then

$$\mathcal{M}(T') = (\mathcal{M}(T))^{t}.$$

3.133 *column rank equals row rank*

Suppose $A \in \mathbf{F}^{m,n}$. Then the column rank of A equals the row rank of A.

5A Invariant Subspaces

5.1 definition: *operator*

A linear map from a vector space to itself is called an *operator*.

Recall that we defined the notation $\mathcal{L}(V)$ to mean $\mathcal{L}(V, V)$.

5.2 definition: *invariant subspace*

Suppose $T \in \mathcal{L}(V)$. A subspace U of V is called *invariant* under T if $Tu \in U$ for every $u \in U$.

5.5 definition: *eigenvalue*

Suppose $T \in \mathcal{L}(V)$. A number $\lambda \in \mathbf{F}$ is called an *eigenvalue* of T if there exists $v \in V$ such that $v \neq 0$ and $Tv = \lambda v$.

5.7 *equivalent conditions to be an eigenvalue*

Suppose V is finite-dimensional, $T \in \mathcal{L}(V)$, and $\lambda \in \mathbf{F}$. Then the following are equivalent.

(a) λ is an eigenvalue of T.

(b) $T - \lambda I$ is not injective.

(c) $T - \lambda I$ is not surjective.

(d) $T - \lambda I$ is not invertible.

Reminder: $I \in \mathcal{L}(V)$ is the identity operator. Thus $Iv = v$ for all $v \in V$.

5.8 definition: *eigenvector*

Suppose $T \in \mathcal{L}(V)$ and $\lambda \in \mathbf{F}$ is an eigenvalue of T. A vector $v \in V$ is called an *eigenvector* of T corresponding to λ if $v \neq 0$ and $Tv = \lambda v$.

5.11 *linearly independent eigenvectors*

Suppose $T \in \mathcal{L}(V)$. Then every list of eigenvectors of T corresponding to distinct eigenvalues of T is linearly independent.

5.12 *operator cannot have more eigenvalues than dimension of vector space*

Suppose V is finite-dimensional. Then each operator on V has at most $\dim V$ distinct eigenvalues.

Polynomials Applied to Operators

5.13 notation: T^m

Suppose $T \in \mathcal{L}(V)$ and m is a positive integer.

- $T^m \in \mathcal{L}(V)$ is defined by $T^m = \underbrace{T \cdots T}_{m \text{ times}}$.
- T^0 is defined to be the identity operator I on V.
- If T is invertible with inverse T^{-1}, then $T^{-m} \in \mathcal{L}(V)$ is defined by

$$T^{-m} = (T^{-1})^m.$$

5.14 notation: $p(T)$

Suppose $T \in \mathcal{L}(V)$ and $p \in \mathcal{P}(\mathbf{F})$ is a polynomial given by

$$p(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_m z^m$$

for all $z \in \mathbf{F}$. Then $p(T)$ is the operator on V defined by

$$p(T) = a_0 I + a_1 T + a_2 T^2 + \cdots + a_m T^m.$$

5.16 definition: *product of polynomials*

If $p, q \in \mathcal{P}(\mathbf{F})$, then $pq \in \mathcal{P}(\mathbf{F})$ is the polynomial defined by

$$(pq)(z) = p(z)q(z)$$

for all $z \in \mathbf{F}$.

5.17 *multiplicative properties*

Suppose $p, q \in \mathcal{P}(\mathbf{F})$ and $T \in \mathcal{L}(V)$. Then

(a) $(pq)(T) = p(T)q(T)$;

(b) $p(T)q(T) = q(T)p(T)$.

Informal proof: When a product of polynomials is expanded using the distributive property, it does not matter whether the symbol is z or T.

5.18 *null space and range of $p(T)$ are invariant under T*

Suppose $T \in \mathcal{L}(V)$ and $p \in \mathcal{P}(\mathbf{F})$. Then null $p(T)$ and range $p(T)$ are invariant under T.

5.19 *existence of eigenvalues*

Every operator on a finite-dimensional nonzero complex vector space has an eigenvalue.

Eigenvalues and the Minimal Polynomial

5.21 definition: *monic polynomial*

A *monic polynomial* is a polynomial whose highest-degree coefficient equals 1.

5.22 *existence, uniqueness, and degree of minimal polynomial*

Suppose V is finite-dimensional and $T \in \mathcal{L}(V)$. Then there is a unique monic polynomial $p \in \mathcal{P}(\mathbf{F})$ of smallest degree such that $p(T) = 0$. Furthermore, $\deg p \leq \dim V$.

5.24 definition: ***minimal polynomial***

Suppose V is finite-dimensional and $T \in \mathcal{L}(V)$. Then the *minimal polynomial* of T is the unique monic polynomial $p \in \mathcal{P}(\mathbf{F})$ of smallest degree such that $p(T) = 0$.

5.27 ***eigenvalues are the zeros of the minimal polynomial***

Suppose V is finite-dimensional and $T \in \mathcal{L}(V)$.

(a) The zeros of the minimal polynomial of T are the eigenvalues of T.

(b) If V is a complex vector space, then the minimal polynomial of T has the form

$$(z - \lambda_1)\cdots(z - \lambda_m),$$

where $\lambda_1, \ldots, \lambda_m$ is a list of all eigenvalues of T, possibly with repetitions.

5.29 *$q(T) = 0 \iff q$ is a polynomial multiple of the minimal polynomial*

Suppose V is finite-dimensional, $T \in \mathcal{L}(V)$, and $q \in \mathcal{P}(\mathbf{F})$. Then $q(T) = 0$ if and only if q is a polynomial multiple of the minimal polynomial of T.

5.31 *minimal polynomial of a restriction operator*

Suppose V is finite-dimensional, $T \in \mathcal{L}(V)$, and U is a subspace of V that is invariant under T. Then the minimal polynomial of T is a polynomial multiple of the minimal polynomial of $T|_U$.

5.32 *T not invertible $\iff$ constant term of minimal polynomial of T is 0*

Suppose V is finite-dimensional and $T \in \mathcal{L}(V)$. Then T is not invertible if and only if the constant term of the minimal polynomial of T is 0.

Eigenvalues on Odd-Dimensional Real Vector Spaces

5.33 *even-dimensional null space*

Suppose $\mathbf{F} = \mathbf{R}$ and V is finite-dimensional. Suppose also that $T \in \mathcal{L}(V)$ and $b, c \in \mathbf{R}$ with $b^2 < 4c$. Then $\dim \operatorname{null}(T^2 + bT + cI)$ is an even number.

5.34 *operators on odd-dimensional vector spaces have eigenvalues*

Every operator on an odd-dimensional vector space has an eigenvalue.

5C Upper-Triangular Matrices

5.35 definition: *matrix of an operator, $\mathcal{M}(T)$*

Suppose $T \in \mathcal{L}(V)$. The *matrix of* T with respect to a basis $v_1, \ldots, v_n$ of V is the n-by-n matrix

$$\mathcal{M}(T) = \begin{pmatrix} A_{1,1} & \cdots & A_{1,n} \\ \vdots & & \vdots \\ A_{n,1} & \cdots & A_{n,n} \end{pmatrix}$$

whose entries $A_{j,k}$ are defined by

$$Tv_k = A_{1,k}v_1 + \cdots + A_{n,k}v_n.$$

The notation $\mathcal{M}(T, (v_1, \ldots, v_n))$ is used if the basis is not clear from the context.

5.37 definition: *diagonal of a matrix*

The *diagonal* of a square matrix consists of the entries on the line from the upper left corner to the bottom right corner.

5.38 definition: *upper-triangular matrix*

A square matrix is called *upper triangular* if all entries below the diagonal are 0.

5.39 *conditions for upper-triangular matrix*

Suppose $T \in \mathcal{L}(V)$ and $v_1, \ldots, v_n$ is a basis of V. Then the following are equivalent.

(a) The matrix of T with respect to $v_1, \ldots, v_n$ is upper triangular.

(b) $\operatorname{span}(v_1, \ldots, v_k)$ is invariant under T for each $k = 1, \ldots, n$.

(c) $Tv_k \in \operatorname{span}(v_1, \ldots, v_k)$ for each $k = 1, \ldots, n$.

5.40 *equation satisfied by operator with upper-triangular matrix*

Suppose $T \in \mathcal{L}(V)$ and V has a basis with respect to which T has an upper-triangular matrix with diagonal entries $\lambda_1, \ldots, \lambda_n$. Then

$$(T - \lambda_1 I) \cdots (T - \lambda_n I) = 0.$$

5.41 *determination of eigenvalues from upper-triangular matrix*

Suppose $T \in \mathcal{L}(V)$ has an upper-triangular matrix with respect to some basis of V. Then the eigenvalues of T are precisely the entries on the diagonal of that upper-triangular matrix.

5.44 *necessary and sufficient condition to have an upper-triangular matrix*

Suppose V is finite-dimensional and $T \in \mathcal{L}(V)$. Then T has an upper-triangular matrix with respect to some basis of V if and only if the minimal polynomial of T equals $(z - \lambda_1)\cdots(z - \lambda_m)$ for some $\lambda_1, \ldots, \lambda_m \in \mathbf{F}$.

5.47 *if* $\mathbf{F} = \mathbf{C}$, *then every operator on* V *has an upper-triangular matrix*

Suppose V is a finite-dimensional complex vector space and $T \in \mathcal{L}(V)$. Then T has an upper-triangular matrix with respect to some basis of V.

5D Diagonalizable Operators

5.48 definition: *diagonal matrix*

A *diagonal matrix* is a square matrix that is 0 everywhere except possibly on the diagonal.

5.50 definition: *diagonalizable*

An operator on V is called *diagonalizable* if the operator has a diagonal matrix with respect to some basis of V.

5.52 definition: *eigenspace,* $E(\lambda, T)$

Suppose $T \in \mathcal{L}(V)$ and $\lambda \in \mathbf{F}$. The *eigenspace* of T corresponding to λ is the subspace $E(\lambda, T)$ of V defined by

$$E(\lambda, T) = \operatorname{null}(T - \lambda I) = \{v \in V : Tv = \lambda v\}.$$

Hence $E(\lambda, T)$ is the set of all eigenvectors of T corresponding to λ, along with the 0 vector.

5.54 *sum of eigenspaces is a direct sum*

Suppose $T \in \mathcal{L}(V)$ and $\lambda_1, \ldots, \lambda_m$ are distinct eigenvalues of T. Then

$$E(\lambda_1, T) + \cdots + E(\lambda_m, T)$$

is a direct sum. Furthermore, if V is finite-dimensional, then

$$\dim E(\lambda_1, T) + \cdots + \dim E(\lambda_m, T) \leq \dim V.$$

5.55 *conditions equivalent to diagonalizability*

Suppose V is finite-dimensional and $T \in \mathcal{L}(V)$. Let $\lambda_1, \dots, \lambda_m$ denote the distinct eigenvalues of T. Then the following are equivalent.

(a) T is diagonalizable.

(b) V has a basis consisting of eigenvectors of T.

(c) $V = E(\lambda_1, T) \oplus \cdots \oplus E(\lambda_m, T)$.

(d) $\dim V = \dim E(\lambda_1, T) + \cdots + \dim E(\lambda_m, T)$.

5.58 *enough eigenvalues implies diagonalizability*

Suppose V is finite-dimensional and $T \in \mathcal{L}(V)$ has $\dim V$ distinct eigenvalues. Then T is diagonalizable.

5.62 *necessary and sufficient condition for diagonalizability*

Suppose V is finite-dimensional and $T \in \mathcal{L}(V)$. Then T is diagonalizable if and only if the minimal polynomial of T equals $(z - \lambda_1)\cdots(z - \lambda_m)$ for some list of distinct numbers $\lambda_1, \ldots, \lambda_m \in \mathbf{F}$.

5.65 *restriction of diagonalizable operator to invariant subspace*

Suppose $T \in \mathcal{L}(V)$ is diagonalizable and U is a subspace of V that is invariant under T. Then $T|_U$ is a diagonalizable operator on U.

5.66 definition: *Gershgorin disks*

Suppose $T \in \mathcal{L}(V)$ and $v_1, \ldots, v_n$ is a basis of V. Let A denote the matrix of T with respect to this basis. A *Gershgorin disk* of T with respect to the basis $v_1, \ldots, v_n$ is a set of the form

$$\Bigl\{z \in \mathbf{F} : |z - A_{j,j}| \leq \sum_{\substack{k=1 \\ k \neq j}}^{n} |A_{j,k}|\Bigr\},$$

where $j \in \{1, \ldots, n\}$.

5.67 *Gershgorin disk theorem*

Suppose $T \in \mathcal{L}(V)$ and $v_1, \ldots, v_n$ is a basis of V. Then each eigenvalue of T is contained in some Gershgorin disk of T with respect to the basis $v_1, \ldots, v_n$.

5E Commuting Operators

5.71 definition: *commute*

- Two operators S and T on the same vector space *commute* if $ST = TS$.
- Two square matrices A and B of the same size *commute* if $AB = BA$.

5.74 *commuting operators correspond to commuting matrices*

Suppose $S, T \in \mathcal{L}(V)$ and $v_1, \ldots, v_n$ is a basis of V. Then S and T commute if and only if $\mathcal{M}(S, (v_1, \ldots, v_n))$ and $\mathcal{M}(T, (v_1, \ldots, v_n))$ commute.

5.75 *eigenspace is invariant under commuting operator*

Suppose $S, T \in \mathcal{L}(V)$ commute and $\lambda \in \mathbf{F}$. Then $E(\lambda, S)$ is invariant under T.

5.76 *simultaneous diagonalizablity $\iff$ commutativity*

Two diagonalizable operators on the same vector space have diagonal matrices with respect to the same basis if and only if the two operators commute.

5.78 *common eigenvector for commuting operators*

Every pair of commuting operators on a finite-dimensional nonzero complex vector space has a common eigenvector.

5.80 *commuting operators are simultaneously upper triangularizable*

Suppose V is a finite-dimensional complex vector space and S, T are commuting operators on V. Then there is a basis of V with respect to which both S and T have upper-triangular matrices.

5.81 *eigenvalues of sum and product of commuting operators*

Suppose V is a finite-dimensional complex vector space and S, T are commuting operators on V. Then

- every eigenvalue of $S + T$ is an eigenvalue of S plus an eigenvalue of T,
- every eigenvalue of ST is an eigenvalue of S times an eigenvalue of T.

6A *Inner Products and Norms*

6.1 definition: *dot product*

For $x, y \in \mathbf{R}^n$, the *dot product* of x and y, denoted by $x \cdot y$, is defined by

$$x \cdot y = x_1 y_1 + \cdots + x_n y_n,$$

where $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$.

6.2 definition: *inner product*

An *inner product* on V is a function that takes each ordered pair (u, v) of elements of V to a number $\langle u, v \rangle \in \mathbf{F}$ and has the following properties.

positivity
$\langle v, v \rangle \geq 0$ for all $v \in V$.

definiteness
$\langle v, v \rangle = 0$ if and only if $v = 0$.

additivity in first slot
$\langle u + v, w \rangle = \langle u, w \rangle + \langle v, w \rangle$ for all $u, v, w \in V$.

homogeneity in first slot
$\langle \lambda u, v \rangle = \lambda \langle u, v \rangle$ for all $\lambda \in \mathbf{F}$ and all $u, v \in V$.

conjugate symmetry
$\langle u, v \rangle = \overline{\langle v, u \rangle}$ for all $u, v \in V$.

6.4 definition: *inner product space*

An *inner product space* is a vector space V along with an inner product on V.

6.6 *basic properties of an inner product*

(a) For each fixed $v \in V$, the function that takes $u \in V$ to $\langle u, v \rangle$ is a linear map from V to $\mathbf{F}$.

(b) $\langle 0, v \rangle = 0$ for every $v \in V$.

(c) $\langle v, 0 \rangle = 0$ for every $v \in V$.

(d) $\langle u, v + w \rangle = \langle u, v \rangle + \langle u, w \rangle$ for all $u, v, w \in V$.

(e) $\langle u, \lambda v \rangle = \overline{\lambda} \langle u, v \rangle$ for all $\lambda \in \mathbf{F}$ and all $u, v \in V$.

6.7 definition: ***norm,*** $\|v\|$

For $v \in V$, the *norm* of v, denoted by $\|v\|$, is defined by

$$\|v\| = \sqrt{\langle v, v \rangle}.$$

6.9 ***basic properties of the norm***

Suppose $v \in V$.

(a) $\|v\| = 0$ if and only if $v = 0$.

(b) $\|\lambda v\| = |\lambda| \, \|v\|$ for all $\lambda \in \mathbf{F}$.

6.10 definition: ***orthogonal***

Two vectors $u, v \in V$ are called *orthogonal* if $\langle u, v \rangle = 0$.

6.11 *orthogonality and* 0

(a) 0 is orthogonal to every vector in V.

(b) 0 is the only vector in V that is orthogonal to itself.

6.12 *Pythagorean theorem*

Suppose $u, v \in V$. If u and v are orthogonal, then

$$\|u + v\|^2 = \|u\|^2 + \|v\|^2.$$

6.13 *an orthogonal decomposition*

Suppose $u, v \in V$, with $v \neq 0$. Set $c = \dfrac{\langle u, v\rangle}{\|v\|^2}$ and $w = u - \dfrac{\langle u, v\rangle}{\|v\|^2}v$. Then

$$u = cv + w \quad \text{and} \quad \langle w, v\rangle = 0.$$

6.14 *Cauchy–Schwarz inequality*

Suppose $u, v \in V$. Then

$$|\langle u, v\rangle| \leq \|u\|\,\|v\|.$$

This inequality is an equality if and only if one of u, v is a scalar multiple of the other.

6.17 *triangle inequality*

Suppose $u, v \in V$. Then

$$\|u + v\| \leq \|u\| + \|v\|.$$

This inequality is an equality if and only if one of u, v is a nonnegative real multiple of the other.

6.21 *parallelogram equality*

Suppose $u, v \in V$. Then

$$\|u + v\|^2 + \|u - v\|^2 = 2(\|u\|^2 + \|v\|^2).$$

6B Orthonormal Bases

Orthonormal Lists and the Gram–Schmidt Procedure

6.22 definition: *orthonormal*

- A list of vectors is called *orthonormal* if each vector in the list has norm 1 and is orthogonal to all the other vectors in the list.
- In other words, a list $e_1, \ldots, e_m$ of vectors in V is orthonormal if

$$\langle e_j, e_k \rangle = \begin{cases} 1 & \text{if } j = k, \\ 0 & \text{if } j \neq k \end{cases}$$

for all $j, k \in \{1, \ldots, m\}$.

6.24 *norm of an orthonormal linear combination*

Suppose $e_1, \ldots, e_m$ is an orthonormal list of vectors in V. Then

$$\|a_1 e_1 + \cdots + a_m e_m\|^2 = |a_1|^2 + \cdots + |a_m|^2$$

for all $a_1, \ldots, a_m \in \mathbf{F}$.

6.25 *orthonormal lists are linearly independent*

Every orthonormal list of vectors is linearly independent.

6.26 *Bessel's inequality*

Suppose $e_1, \dots, e_m$ is an orthonormal list of vectors in V. If $v \in V$ then

$$|\langle v, e_1 \rangle|^2 + \cdots + |\langle v, e_m \rangle|^2 \leq \|v\|^2.$$

6.27 definition: *orthonormal basis*

An *orthonormal basis* of V is an orthonormal list of vectors in V that is also a basis of V.

6.28 *orthonormal lists of the right length are orthonormal bases*

Suppose V is finite-dimensional. Then every orthonormal list of vectors in V of length $\dim V$ is an orthonormal basis of V.

6.30 *writing a vector as a linear combination of an orthonormal basis*

Suppose $e_1, \ldots, e_n$ is an orthonormal basis of V and $u, v \in V$. Then

(a) $v = \langle v, e_1 \rangle e_1 + \cdots + \langle v, e_n \rangle e_n$,

(b) $\|v\|^2 = |\langle v, e_1 \rangle|^2 + \cdots + |\langle v, e_n \rangle|^2$,

(c) $\langle u, v \rangle = \langle u, e_1 \rangle \overline{\langle v, e_1 \rangle} + \cdots + \langle u, e_n \rangle \overline{\langle v, e_n \rangle}$.

6.32 *Gram–Schmidt procedure*

Suppose $v_1, \ldots, v_m$ is a linearly independent list of vectors in V. Let $f_1 = v_1$. For $k = 2, \ldots, m$, define f_k inductively by

$$f_k = v_k - \frac{\langle v_k, f_1 \rangle}{\|f_1\|^2} f_1 - \cdots - \frac{\langle v_k, f_{k-1} \rangle}{\|f_{k-1}\|^2} f_{k-1}.$$

For each $k = 1, \ldots, m$, let $e_k = \dfrac{f_k}{\|f_k\|}$. Then $e_1, \ldots, e_m$ is an orthonormal list of vectors in V such that

$$\operatorname{span}(v_1, \ldots, v_k) = \operatorname{span}(e_1, \ldots, e_k)$$

for each $k = 1, \ldots, m$.

6.35 *existence of orthonormal basis*

Every finite-dimensional inner product space has an orthonormal basis.

6.36 *every orthonormal list extends to an orthonormal basis*

Suppose V is finite-dimensional. Then every orthonormal list of vectors in V can be extended to an orthonormal basis of V.

6.37 *upper-triangular matrix with respect to some orthonormal basis*

Suppose V is finite-dimensional and $T \in \mathcal{L}(V)$. Then T has an upper-triangular matrix with respect to some orthonormal basis of V if and only if the minimal polynomial of T equals $(z - \lambda_1)\cdots(z - \lambda_m)$ for some $\lambda_1, \ldots, \lambda_m \in \mathbf{F}$.

6.38 *Schur's theorem*

Every operator on a finite-dimensional complex inner product space has an upper-triangular matrix with respect to some orthonormal basis.

Linear Functionals on Inner Product Spaces

6.42 *Riesz representation theorem*

Suppose V is finite-dimensional and φ is a linear functional on V. Then there is a unique vector $v \in V$ such that

$$\varphi(u) = \langle u, v \rangle$$

for every $u \in V$.

6.46 definition: *orthogonal complement, $U^\perp$*

If U is a subset of V, then the *orthogonal complement* of U, denoted by $U^\perp$, is the set of all vectors in V that are orthogonal to every vector in U:

$$U^\perp = \{v \in V : \langle u, v\rangle = 0 \text{ for every } u \in U\}.$$

6.48 *properties of orthogonal complement*

(a) If U is a subset of V, then $U^\perp$ is a subspace of V.

(b) $\{0\}^\perp = V$.

(c) $V^\perp = \{0\}$.

(d) If U is a subset of V, then $U \cap U^\perp \subseteq \{0\}$.

(e) If G and H are subsets of V and $G \subseteq H$, then $H^\perp \subseteq G^\perp$.

6.49 *direct sum of a subspace and its orthogonal complement*

Suppose U is a finite-dimensional subspace of V. Then

$$V = U \oplus U^{\perp}.$$

6.51 *dimension of orthogonal complement*

Suppose V is finite-dimensional and U is a subspace of V. Then

$$\dim U^{\perp} = \dim V - \dim U.$$

6.52 *orthogonal complement of the orthogonal complement*

Suppose U is a finite-dimensional subspace of V. Then

$$U = (U^\perp)^\perp.$$

6.54 $U^\perp = \{0\} \iff U = V$ *(for U a finite-dimensional subspace of* V*)*

Suppose U is a finite-dimensional subspace of V. Then

$$U^\perp = \{0\} \iff U = V.$$

6.55 definition: *orthogonal projection,* P_U

Suppose U is a finite-dimensional subspace of V. The *orthogonal projection* of V onto U is the operator $P_U \in \mathcal{L}(V)$ defined as follows: For each $v \in V$, write $v = u + w$, where $u \in U$ and $w \in U^{\perp}$. Then let $P_U v = u$.

6.57 *properties of orthogonal projection* P_U

Suppose U is a finite-dimensional subspace of V. Then

(a) $P_U \in \mathcal{L}(V)$;

(b) $P_U u = u$ for every $u \in U$;

(c) $P_U w = 0$ for every $w \in U^{\perp}$;

(d) range $P_U = U$;

(e) null $P_U = U^{\perp}$;

(f) $v - P_U v \in U^{\perp}$ for every $v \in V$;

(g) $P_U^{\,2} = P_U$;

(h) $\|P_U v\| \leq \|v\|$ for every $v \in V$;

(i) if $e_1, \ldots, e_m$ is an orthonormal basis of U and $v \in V$, then

$$P_U v = \langle v, e_1 \rangle e_1 + \cdots + \langle v, e_m \rangle e_m.$$

6.58 *Riesz representation theorem, revisited*

Suppose V is finite-dimensional. For each $v \in V$, define $\varphi_v \in V'$ by

$$\varphi_v(u) = \langle u, v \rangle$$

for each $u \in V$. Then $v \mapsto \varphi_v$ is a one-to-one function from V onto V'.

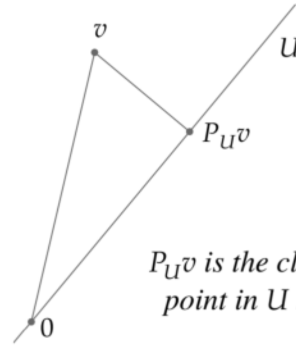
Minimization Problems

6.61 *minimizing distance to a subspace*

Suppose U is a finite-dimensional subspace of V, $v \in V$, and $u \in U$. Then

$$\|v - P_U v\| \leq \|v - u\|.$$

Furthermore, the inequality above is an equality if and only if $u = P_U v$.



$P_U v$ is the closest point in U to v.

Pseudoinverse

6.67 *restriction of a linear map to obtain a one-to-one and onto map*

Suppose V is finite-dimensional and $T \in \mathcal{L}(V, W)$. Then $T|_{(\operatorname{null} T)^{\perp}}$ is a one-to-one map of $(\operatorname{null} T)^{\perp}$ onto range T.

6.68 definition: *pseudoinverse, $T^{\dagger}$*

Suppose that V is finite-dimensional and $T \in \mathcal{L}(V, W)$. The *pseudoinverse* $T^{\dagger} \in \mathcal{L}(W, V)$ of T is the linear map from W to V defined by

$$T^{\dagger} w = (T|_{(\operatorname{null} T)^{\perp}})^{-1} P_{\operatorname{range} T}\, w$$

for each $w \in W$.

6.69 *algebraic properties of the pseudoinverse*

Suppose V is finite-dimensional and $T \in \mathcal{L}(V, W)$.

(a) If T is invertible, then $T^\dagger = T^{-1}$.

(b) $TT^\dagger = P_{\operatorname{range} T}$ = the orthogonal projection of W onto range T.

(c) $T^\dagger T = P_{(\operatorname{null} T)^\perp}$ = the orthogonal projection of V onto $(\operatorname{null} T)^\perp$.

6.70 *pseudoinverse provides best approximate solution or best solution*

Suppose V is finite-dimensional, $T \in \mathcal{L}(V, W)$, and $b \in W$.

(a) If $x \in V$, then

$$\|T(T^\dagger b) - b\| \leq \|Tx - b\|,$$

with equality if and only if $x \in T^\dagger b + \operatorname{null} T$.

(b) If $x \in T^\dagger b + \operatorname{null} T$, then

$$\|T^\dagger b\| \leq \|x\|,$$

with equality if and only if $x = T^\dagger b$.

7A Self-Adjoint and Normal Operators

Adjoints

7.1 definition: *adjoint*, T^*

Suppose $T \in \mathcal{L}(V, W)$. The *adjoint* of T is the function $T^* \colon W \to V$ such that

$$\langle Tv, w \rangle = \langle v, T^* w \rangle$$

for every $v \in V$ and every $w \in W$.

7.4 *adjoint of a linear map is a linear map*

If $T \in \mathcal{L}(V, W)$, then $T^* \in \mathcal{L}(W, V)$.

7.5 *properties of the adjoint*

Suppose $T \in \mathcal{L}(V, W)$. Then

(a) $(S + T)^* = S^* + T^*$ for all $S \in \mathcal{L}(V, W)$;

(b) $(\lambda T)^* = \overline{\lambda} T^*$ for all $\lambda \in \mathbf{F}$;

(c) $(T^*)^* = T$;

(d) $(ST)^* = T^* S^*$ for all $S \in \mathcal{L}(W, U)$ (here U is a finite-dimensional inner product space over $\mathbf{F}$);

(e) $I^* = I$, where I is the identity operator on V;

(f) if T is invertible, then T^* is invertible and $(T^*)^{-1} = (T^{-1})^*$.

7.6 *null space and range of T^**

Suppose $T \in \mathcal{L}(V, W)$. Then

(a) $\operatorname{null} T^* = (\operatorname{range} T)^{\perp}$;

(b) $\operatorname{range} T^* = (\operatorname{null} T)^{\perp}$;

(c) $\operatorname{null} T = (\operatorname{range} T^*)^{\perp}$;

(d) $\operatorname{range} T = (\operatorname{null} T^*)^{\perp}$.

7.7 definition: *conjugate transpose, A^**

The *conjugate transpose* of an m-by-n matrix A is the n-by-m matrix A^* obtained by interchanging the rows and columns and then taking the complex conjugate of each entry. In other words, if $j \in \{1, \ldots, n\}$ and $k \in \{1, \ldots, m\}$, then

$$(A^*)_{j,k} = \overline{A_{k,j}}.$$

7.9 *matrix of T^* equals conjugate transpose of matrix of T*

Let $T \in \mathcal{L}(V, W)$. Suppose $e_1, \ldots, e_n$ is an orthonormal basis of V and $f_1, \ldots, f_m$ is an orthonormal basis of W. Then $\mathcal{M}(T^*, (f_1, \ldots, f_m), (e_1, \ldots, e_n))$ is the conjugate transpose of $\mathcal{M}(T, (e_1, \ldots, e_n), (f_1, \ldots, f_m))$. In other words,

$$\mathcal{M}(T^*) = (\mathcal{M}(T))^*.$$

Self-Adjoint Operators

7.10 definition: *self-adjoint*

An operator $T \in \mathcal{L}(V)$ is called *self-adjoint* if $T = T^*$.

An operator $T \in \mathcal{L}(V)$ is self-adjoint if and only if

$$\langle Tv, w \rangle = \langle v, Tw \rangle$$

for all $v, w \in V$.

7.12 *eigenvalues of self-adjoint operators*

Every eigenvalue of a self-adjoint operator is real.

7.13 *Tv is orthogonal to v for all $v \iff T = 0$ (assuming $\mathbf{F} = \mathbf{C}$)*

Suppose V is a complex inner product space and $T \in \mathcal{L}(V)$. Then

$$\langle Tv, v \rangle = 0 \text{ for every } v \in V \iff T = 0.$$

7.14 *$\langle Tv, v\rangle$ is real for all $v \iff T$ is self-adjoint (assuming $\mathbf{F} = \mathbf{C}$)*

Suppose V is a complex inner product space and $T \in \mathcal{L}(V)$. Then

$$T \text{ is self-adjoint} \iff \langle Tv, v\rangle \in \mathbf{R} \text{ for every } v \in V.$$

7.16 *T self-adjoint and $\langle Tv, v\rangle = 0$ for all $v \iff T = 0$*

Suppose T is a self-adjoint operator on V. Then

$$\langle Tv, v\rangle = 0 \text{ for every } v \in V \iff T = 0.$$

Normal Operators

7.18 definition: *normal*

- An operator on an inner product space is called *normal* if it commutes with its adjoint.
- In other words, $T \in \mathcal{L}(V)$ is normal if $TT^* = T^*T$.

7.20 *T is normal if and only if Tv and T^*v have the same norm*

Suppose $T \in \mathcal{L}(V)$. Then

$$T \text{ is normal} \iff \|Tv\| = \|T^*v\| \text{ for every } v \in V.$$

7.21 *range, null space, and eigenvectors of a normal operator*

Suppose $T \in \mathcal{L}(V)$ is normal. Then

(a) $\operatorname{null} T = \operatorname{null} T^*$;

(b) $\operatorname{range} T = \operatorname{range} T^*$;

(c) $V = \operatorname{null} T \oplus \operatorname{range} T$;

(d) $T - \lambda I$ is normal for every $\lambda \in \mathbf{F}$;

(e) if $v \in V$ and $\lambda \in \mathbf{F}$, then $Tv = \lambda v$ if and only if $T^* v = \overline{\lambda} v$.

7.22 *orthogonal eigenvectors for normal operators*

Suppose $T \in \mathcal{L}(V)$ is normal. Then eigenvectors of T corresponding to distinct eigenvalues are orthogonal.

7.23 *T is normal $\iff$ the real and imaginary parts of T commute*

Suppose $\mathbf{F} = \mathbf{C}$ and $T \in \mathcal{L}(V)$. Then T is normal if and only if there exist commuting self-adjoint operators A and B such that $T = A + iB$.

7B Spectral Theorem

Real Spectral Theorem

7.26 *invertible quadratic expressions*

Suppose $T \in \mathcal{L}(V)$ is self-adjoint and $b, c \in \mathbf{R}$ are such that $b^2 < 4c$. Then

$$T^2 + bT + cI$$

is an invertible operator.

7.27 *minimal polynomial of self-adjoint operator*

Suppose $T \in \mathcal{L}(V)$ is self-adjoint. Then the minimal polynomial of T equals $(z - \lambda_1)\cdots(z - \lambda_m)$ for some $\lambda_1, \ldots, \lambda_m \in \mathbf{R}$.

7.29 *real spectral theorem*

Suppose $\mathbf{F} = \mathbf{R}$ and $T \in \mathcal{L}(V)$. Then the following are equivalent.

(a) T is self-adjoint.

(b) T has a diagonal matrix with respect to some orthonormal basis of V.

(c) V has an orthonormal basis consisting of eigenvectors of T.

Complex Spectral Theorem

7.31 *complex spectral theorem*

Suppose $\mathbf{F} = \mathbf{C}$ and $T \in \mathcal{L}(V)$. Then the following are equivalent.

(a) T is normal.

(b) T has a diagonal matrix with respect to some orthonormal basis of V.

(c) V has an orthonormal basis consisting of eigenvectors of T.

7C Positive Operators

7.34 definition: *positive operator*

An operator $T \in \mathcal{L}(V)$ is called *positive* if T is self-adjoint and

$$\langle Tv, v \rangle \geq 0$$

for all $v \in V$.

7.36 definition: *square root*

An operator R is called a *square root* of an operator T if $R^2 = T$.

7.38 *characterization of positive operators*

Let $T \in \mathcal{L}(V)$. Then the following are equivalent.

(a) T is a positive operator.

(b) T is self-adjoint and all eigenvalues of T are nonnegative.

(c) With respect to some orthonormal basis of V, the matrix of T is a diagonal matrix with only nonnegative numbers on the diagonal.

(d) T has a positive square root.

(e) T has a self-adjoint square root.

(f) $T = R^*R$ for some $R \in \mathcal{L}(V)$.

7.39 *each positive operator has only one positive square root*

Every positive operator on V has a unique positive square root.

7.40 notation: $\sqrt{T}$

For T a positive operator, $\sqrt{T}$ denotes the unique positive square root of T.

7.43 *T positive and $\langle Tv, v\rangle = 0 \implies Tv = 0$*

Suppose T is a positive operator on V and $v \in V$ is such that $\langle Tv, v\rangle = 0$. Then $Tv = 0$.

7D Isometries, Unitary Operators, and Matrix Factorization

Isometries

7.44 definition: *isometry*

A linear map $S \in \mathcal{L}(V, W)$ is called an *isometry* if

$$\|Sv\| = \|v\|$$

for every $v \in V$. In other words, a linear map is an isometry if it preserves norms.

7.49 *characterization of isometries*

Suppose $S \in \mathcal{L}(V, W)$. Suppose $e_1, \ldots, e_n$ is an orthonormal basis of V and $f_1, \ldots, f_m$ is an orthonormal basis of W. Then the following are equivalent.

(a) S is an isometry.

(b) $S^*S = I$.

(c) $\langle Su, Sv \rangle = \langle u, v \rangle$ for all $u, v \in V$.

(d) $Se_1, \ldots, Se_n$ is an orthonormal list in W.

(e) The columns of $\mathcal{M}(S, (e_1, \ldots, e_n), (f_1, \ldots, f_m))$ form an orthonormal list in $\mathbf{F}^m$ with respect to the Euclidean inner product.

Unitary Operators

7.51 definition: *unitary operator*

An operator $S \in \mathcal{L}(V)$ is called *unitary* if S is an invertible isometry.

7.53 *characterization of unitary operators*

Suppose $S \in \mathcal{L}(V)$. Suppose $e_1, \ldots, e_n$ is an orthonormal basis of V. Then the following are equivalent.

(a) S is a unitary operator.

(b) $S^*S = SS^* = I$.

(c) S is invertible and $S^{-1} = S^*$.

(d) $Se_1, \ldots, Se_n$ is an orthonormal basis of V.

(e) The rows of $\mathcal{M}(S, (e_1, \ldots, e_n))$ form an orthonormal basis of $\mathbf{F}^n$ with respect to the Euclidean inner product.

(f) S^* is a unitary operator.

7.54 *eigenvalues of unitary operators have absolute value* 1

Suppose λ is an eigenvalue of a unitary operator. Then $|\lambda| = 1$.

7.55 *description of unitary operators on complex inner product spaces*

Suppose $\mathbf{F} = \mathbf{C}$ and $S \in \mathcal{L}(V)$. Then the following are equivalent.

(a) S is a unitary operator.

(b) There is an orthonormal basis of V consisting of eigenvectors of S whose corresponding eigenvalues all have absolute value 1.

QR Factorization

7.56 definition: *unitary matrix*

An n-by-n matrix is called *unitary* if its columns form an orthonormal list in $\mathbf{F}^n$.

7.57 *characterizations of unitary matrices*

Suppose Q is an n-by-n matrix. Then the following are equivalent.

(a) Q is a unitary matrix.

(b) The rows of Q form an orthonormal list in $\mathbf{F}^n$.

(c) $\|Qv\| = \|v\|$ for every $v \in \mathbf{F}^n$.

(d) $Q^*Q = QQ^* = I$, the n-by-n matrix with 1's on the diagonal and 0's elsewhere.

7.58 *QR factorization*

Suppose A is a square matrix with linearly independent columns. Then there exist unique matrices Q and R such that Q is unitary, R is upper triangular with only positive numbers on its diagonal, and

$$A = QR.$$

7E Singular Value Decomposition

7.64 *properties of T^*T*

Suppose $T \in \mathcal{L}(V, W)$. Then

(a) T^*T is a positive operator on V;

(b) $\operatorname{null} T^*T = \operatorname{null} T$;

(c) $\operatorname{range} T^*T = \operatorname{range} T^*$;

(d) $\dim \operatorname{range} T = \dim \operatorname{range} T^* = \dim \operatorname{range} T^*T$.

7.65 definition: *singular values*

Suppose $T \in \mathcal{L}(V, W)$. The *singular values* of T are the nonnegative square roots of the eigenvalues of T^*T, listed in decreasing order, each included as many times as the dimension of the corresponding eigenspace of T^*T.

7.68 *role of positive singular values*

Suppose that $T \in \mathcal{L}(V, W)$. Then

(a) T is injective $\iff$ 0 is not a singular value of T;

(b) the number of positive singular values of T equals $\dim \operatorname{range} T$;

(c) T is surjective $\iff$ number of positive singular values of T equals $\dim W$.

list of eigenvalues	**list of singular values**
context: vector spaces	context: inner product spaces
defined only for linear maps from a vector space to itself	defined for linear maps from an inner product space to a possibly different inner product space
can be arbitrary real numbers (if $\mathbf{F} = \mathbf{R}$) or complex numbers (if $\mathbf{F} = \mathbf{C}$)	are nonnegative numbers
can be the empty list if $\mathbf{F} = \mathbf{R}$	length of list equals dimension of domain
includes 0 $\iff$ operator is not invertible	includes 0 $\iff$ linear map is not injective
no standard order, especially if $\mathbf{F} = \mathbf{C}$	always listed in decreasing order

7.69 *isometries characterized by having all singular values equal* 1

Suppose that $S \in \mathcal{L}(V, W)$. Then

$$S \text{ is an isometry} \iff \text{all singular values of } S \text{ equal } 1.$$

7.70 *singular value decomposition*

Suppose $T \in \mathcal{L}(V, W)$ and the positive singular values of T are $s_1, \ldots, s_m$. Then there exist orthonormal lists $e_1, \ldots, e_m$ in V and $f_1, \ldots, f_m$ in W such that

$$Tv = s_1 \langle v, e_1 \rangle f_1 + \cdots + s_m \langle v, e_m \rangle f_m \tag{7.71}$$

for every $v \in V$.

spectral theorem	singular value decomposition
describes only self-adjoint operators (when $\mathbf{F} = \mathbf{R}$) or normal operators (when $\mathbf{F} = \mathbf{C}$)	describes arbitrary linear maps from an inner product space to a possibly different inner product space
produces a single orthonormal basis	produces two orthonormal lists, one for domain space and one for range space, that are not necessarily the same even when range space equals domain space
different proofs depending on whether $\mathbf{F} = \mathbf{R}$ or $\mathbf{F} = \mathbf{C}$	same proof works regardless of whether $\mathbf{F} = \mathbf{R}$ or $\mathbf{F} = \mathbf{C}$

7.75 *singular value decomposition of adjoint and pseudoinverse*

Suppose $T \in \mathcal{L}(V, W)$ and the positive singular values of T are $s_1, \dots, s_m$. Suppose $e_1, \dots, e_m$ and $f_1, \dots, f_m$ are orthonormal lists in V and W such that

7.76
$$Tv = s_1 \langle v, e_1 \rangle f_1 + \cdots + s_m \langle v, e_m \rangle f_m$$

for every $v \in V$. Then

7.77
$$T^* w = s_1 \langle w, f_1 \rangle e_1 + \cdots + s_m \langle w, f_m \rangle e_m$$

and

7.78
$$T^\dagger w = \frac{\langle w, f_1 \rangle}{s_1} e_1 + \cdots + \frac{\langle w, f_m \rangle}{s_m} e_m$$

for every $w \in W$.

7.80 *matrix version of SVD*

Suppose A is an M-by-n matrix of rank $m \geq 1$. Then there exist an M-by-m matrix B with orthonormal columns, an m-by-m diagonal matrix D with positive numbers on the diagonal, and an n-by-m matrix C with orthonormal columns such that

$$A = BDC^*.$$

8D Trace: A Connection Between Matrices and Operators

8.47 definition: *trace of a matrix*

Suppose A is a square matrix with entries in $\mathbf{F}$. The *trace* of A, denoted by $\operatorname{tr} A$, is defined to be the sum of the diagonal entries of A.

8.49 *trace of AB equals trace of BA*

Suppose A is an m-by-n matrix and B is an n-by-m matrix. Then

$$\operatorname{tr}(AB) = \operatorname{tr}(BA).$$

8.50 *trace of matrix of operator does not depend on basis*

Suppose $T \in \mathcal{L}(V)$. Suppose $u_1, \ldots, u_n$ and $v_1, \ldots, v_n$ are bases of V. Then

$$\operatorname{tr} \mathcal{M}(T, (u_1, \ldots, u_n)) = \operatorname{tr} \mathcal{M}(T, (v_1, \ldots, v_n)).$$

8.51 definition: *trace of an operator*

Suppose $T \in \mathcal{L}(V)$. The *trace* of T, denote $\operatorname{tr} T$, is defined by

$$\operatorname{tr} T = \operatorname{tr} \mathcal{M}(T, (v_1, \ldots, v_n)),$$

where $v_1, \ldots, v_n$ is any basis of V.

8.52 *on complex vector spaces, trace equals sum of eigenvalues*

Suppose $\mathbf{F} = \mathbf{C}$ and $T \in \mathcal{L}(V)$. Then $\operatorname{tr} T$ equals the sum of the eigenvalues of T, with each eigenvalue included as many times as its multiplicity.

8.54 *trace and characteristic polynomial*

Suppose $\mathbf{F} = \mathbf{C}$ and $T \in \mathcal{L}(V)$. Let $n = \dim V$. Then $\operatorname{tr} T$ equals the negative of the coefficient of z^{n-1} in the characteristic polynomial of T.

9C Determinants

Defining the Determinant