HOMEWORK 7

CHAPTER 7: INTRODUCTION TO CARDINALITY

7.1. Show that if $A \subset B$, then $A \preceq B$.

7.2. Prove that the relation $\prec$ is transitive.

7.3. Prove that a subset of a finite set is finite.

7.4. Let A and B be finite sets. Prove the following:

(i) $A \cap B$ is finite.

(ii) $A \cup B$ is finite.

7.5. Let A be a finite set and let B be a proper subset of A.

(i) Prove that $B \preceq A$.

(ii) Prove that A and B are not equinumerous.

7.6. Prove that a countable set with an infinite subset must be denumerable.

7.7. Let $\mathbb{F} = \{\frac{a}{b} \mid a, b \in \mathbb{N}\}$ be the set of fractions whose numerators and denominators are natural numbers and define $f : \mathbb{F} \to \mathbb{N} \times \mathbb{N}$ by $f\left(\frac{a}{b}\right) = (a, b)$. Show that f is a bijection.

7.8. Let A and B be nonempty sets. Prove that there is an injection $f : A \to B$ if and only if there is a surjection $g : B \to A$.

7.9. Prove that each of the following sets is denumerable.

(i) The set of nonnegative integers $\mathbb{N} \cup \{0\}$.

(ii) The set of integers $\mathbb{Z}$.

7.10. Prove the following.

(i) The union of two countable sets is countable.

(ii) The union of finitely many countable sets is countable.

7.11. Let A and B be denumerable sets. Prove that the set $A \times B$ is denumerable.

7.12. Let A, B, and C be sets. Define $A \times B \times C = \{(a, b, c) \mid a \in A \text{ and } b \in B \text{ and } c \in C\}$. Prove that $A \times B \times C \equiv (A \times B) \times C$.

7.13. For each natural number n, let $\mathbb{N}^n$ denote the set of ordered n-tuples of natural numbers, so $\mathbb{N}^3 = \mathbb{N} \times \mathbb{N} \times \mathbb{N}$, etc. Prove that $\mathbb{N}^n$ is countable.

7.14. Let $\mathbb{S}$ denote the set of sequences of 0's and 1's, so a typical element of $\mathbb{S}$ looks like $(x_1, x_2, x_3, \ldots)$ where each x_i is either 0 or 1. Prove that $\mathbb{S}$ is uncountable.