HOMEWORK 7

CHAPTER 7: INTRODUCTION TO CARDINALITY

7.1. Show that if $A \subset B$, then $A \preceq B$.

Suppose $A \subset B$. By definition, $A \preceq B$ if there exists an injection $f : A \to B$. So, let us produce such an injection by considering the inclusion map $\iota : A \to B$ defined by $\iota(a) = a$ for all $a \in A$. Since $\iota(a_1) = \iota(a_2)$ implies $\iota(a_1) = a_1 = a_2 = \iota(a_2)$, the inclusion is injective. Hence, $A \preceq B$ whenever $A \subseteq B$.

7.2. Prove that the relation $\prec$ is transitive.

Recall that $A \prec B$ if $A \preceq B$ and $B \npreceq A$. Suppose $A \prec B$ and $B \prec C$. Then there are injections $f : A \to B$ and $g : B \to C$. By composing, we obtain an injection $g \circ f : A \to C$, so $A \preceq C$. All that remains is to show that $C \npreceq A$. Suppose, for contradiction, that there is an injection $h : C \to A$. Then, composing, we have an injection $h \circ f : C \to B$, so $C \preceq B$. But we already know $B \preceq C$, hence $B \preceq C$ and $C \preceq B$. By the Cantor–Bernstein theorem, this implies $B \equiv C$, contradicting $B \prec C$ (which, recall, says $B \preceq C$ and $B \npreceq C$). Therefore, such an injection must not exist, and we may conclude that $A \prec C$ whenever $A \prec B$ and $B \prec C$. That is, $\prec$ is transitive.

7.3. Prove that a subset of a finite set is finite.

Let A be finite and let $B \subset A$. If $B = \varnothing$, then B is finite. If not, since A is finite, there exists $n \in \mathbb{N}$ and a bijection $f : \{1, 2, \ldots, n\} \to A$. So, let us set $C = f^{-1}(B) = \{k \in \{1, \ldots, n\} : f(k) \in B\}$. Since C is a subset of the finite set $\{1, \ldots, n\}$, it is also finite. The map $f|_C : C \to B$ defined by $f|_C(x) = f(x)$ for all $x \in C$ is a bijection (since it is just the restriction of a bijection to the preimage of B under f), so $B \equiv C$. Since C is finite, B is finite. Therefore, we may conclude that every subset of a finite set is finite.

7.4. Let A and B be finite sets. Prove the following:

(i) $A \cap B$ is finite.

(ii) $A \cup B$ is finite.

(i) Since $A \cap B \subset A$ and A is finite, Exercise 7.3 implies $A \cap B$ is finite.

(ii) First note that $B \setminus A \subset B$, so $B \setminus A$ is finite by Exercise 7.3. The sets A and $B \setminus A$ are disjoint and both finite. There are bijections

$$f : \{1, \dots, m\} \to A, \qquad g : \{1, \dots, n\} \to B \setminus A$$

for some $m, n \in \mathbb{N}$. Define

$$h : \{1, \dots, m+n\} \to A \cup (B \setminus A) = A \cup B$$

by

$$h(k) = \begin{cases} f(k), & 1 \leq k \leq m, \\ g(k-m), & m < k \leq m+n. \end{cases}$$

This map is bijective, so $A \cup B$ is finite.

7.5. Let A be a finite set and let B be a proper subset of A.

(i) Prove that $B \preceq A$.

(ii) Prove that A and B are not equinumerous.

(i) Note that, since $B \subset A$, the inclusion map $\iota : B \to A$ defined by $\iota(b) = b$ for all $b \in B$, is an injection. Thus, $B \preceq A$.

(ii) First, note that since A is finite, there is $n \in \mathbb{N}$ such that $A \equiv \mathbb{N}_n$. Since B is a proper subset of A, there exists $a_0 \in A \setminus B$. Let $f : A \to \mathbb{N}_n$ be a bijection. Notice that $f|_B : B \to \mathbb{N}_n$ is an injection which is not a surjection, since $f(a_0) \notin f|_B(B)$. Accordingly, $B \preceq \mathbb{N}_{n-1}$, which implies that $B \not\equiv \mathbb{N}_n$. By transitivity and the fact that $A \equiv \mathbb{N}_n$, we get $B \not\equiv A$.

7.6. Prove that a countable set with an infinite subset must be denumerable.

By definition, a set is *countable* if it is either finite or denumerable. Let A be countable and suppose A has an infinite subset S. If A were finite, every subset of A would also be finite, contradicting the fact that S is infinite. Therefore A cannot be finite. Hence, the only remaining possibility is that A is denumerable.

7.7. Let $\mathbb{F} = \{\frac{a}{b} \mid a, b \in \mathbb{N}\}$ be the set of fractions whose numerators and denominators are natural numbers and define $f : \mathbb{F} \to \mathbb{N} \times \mathbb{N}$ by $f\left(\frac{a}{b}\right) = (a, b)$. Show that f is a bijection.

We interpret $\mathbb{F}$ as the set of formal fractions $\frac{a}{b}$ with $a, b \in \mathbb{N}$ (so different pairs give different elements, even if they represent the same rational number).

First, we show f is injective. To do so, suppose $f\left(\frac{a}{b}\right) = f\left(\frac{c}{d}\right)$. Then, $(a, b) = (c, d)$, so $a = c$ and $b = d$, and therefore $\frac{a}{b} = \frac{c}{d}$ as elements of $\mathbb{F}$. Thus, f is injective. Now, we show f is surjective. Let $(m, n) \in \mathbb{N} \times \mathbb{N}$. Then $\frac{m}{n} \in \mathbb{F}$, and $f\left(\frac{m}{n}\right) = (m, n)$. That is, every element of $\mathbb{N} \times \mathbb{N}$ is in the image of f, so f is surjective. As we have shown that f is both injective and surjective, it is a bijection.

7.8. Let A and B be nonempty sets. Prove that there is an injection $f : A \to B$ if and only if there is a surjection $g : B \to A$.

In the forward direction, suppose $f : A \to B$ is injective. Since $A \neq \varnothing$, choose some fixed $a_0 \in A$. Define $g : B \to A$ by

$$g(b) = \begin{cases} a, & \text{if } b = f(a) \text{ for some } a \in A, \\ a_0, & \text{if } b \notin f(A). \end{cases}$$

Because f is injective, for each $b \in f(A)$ there is a unique a with $f(a) = b$, so g is well defined. Additionally, for any $a \in A$, we have $f(a) \in B$ and $g(f(a)) = a$ by construction, so every $a \in A$ lies in the image of B under g. That is, g is surjective.

Now, in the reverse direction, suppose $g : B \to A$ is surjective. For each $a \in A$, choose one element $b_a \in B$ such that $g(b_a) = a$ (this uses a choice of representative from each nonempty 'fiber' of g, i.e. we are taking on element from each preimage $g^{-1}(\{a\}) = \{b \in B \mid g(b) = a\}$, all of which are nonempty by the surjectivity of g). Let us define $f : A \to B$ by $f(a) = b_a$ for these specified elements b_a. If $f(a_1) = f(a_2)$, then $b_{a_1} = b_{a_2}$ and applying g yields

$$a_1 = g(b_{a_1}) = g(b_{a_2}) = a_2.$$

That is, $a_1 = a_2$ and f is injective.

7.9. Prove that each of the following sets is denumerable.

(i) The set of nonnegative integers $\mathbb{N} \cup \{0\}$.

(ii) The set of integers $\mathbb{Z}$.

(i) Define $g : \mathbb{N} \to \mathbb{N} \cup \{0\}$ by $g(n) = n - 1$ for all $n \in \mathbb{N}$. Then, g is a bijection (to see this, observe that $g(m + 1) = m$ for all $m \in \mathbb{N} \cup \{0\}$ and that $g(n) = g(m)$ is equivalent to $n - 1 = m - 1$, which says that $n = m$). Accordingly, $\mathbb{N} \cup \{0\}$ is denumerable.

(ii) Define $h : \mathbb{N} \to \mathbb{Z}$ by

$$h(n) = \begin{cases} 0, & n = 1, \\ \dfrac{n}{2}, & n \text{ even}, \\ -\dfrac{n-1}{2}, & n \text{ odd and } n \geq 3. \end{cases}$$

We show that every integer appears exactly once in this list, which yields that h is a bijection demonstrates that $\mathbb{Z}$ is denumerable. To see this, observe that if $n \in \mathbb{Z}$ is positive, then $h(2n) = n$. Likewise, if $n \in \mathbb{N}$ is negative, then $h(2(-n) + 1) = -\frac{(2(-n)+1)-1}{2} = -\frac{2(-n)}{2} = n$. So, h is surjective. To see that h is injective, observe that $-\frac{n-1}{2} \neq \frac{m}{2}$ for all $n, m \in \mathbb{N}$ with $n \geq 3$ (as one is negative and the other positive), so we need only observe that

- $\frac{n}{2} = \frac{m}{2} \Rightarrow n = m$
- $-\frac{n-1}{2} = -\frac{m-1}{2} \Rightarrow n - 1 = m - 1 \Rightarrow n = m$

Thus, h is also injective and we have shown $\mathbb{Z} \equiv \mathbb{N}$.

7.10. Prove the following.

(i) The union of two countable sets is countable.

(ii) The union of finitely many countable sets is countable.

(i) Let A and B be countable. There are two cases for each A and B: they are either finite or denumerable. If both are finite, then $A \cup B$ is finite, hence countable. If at least one is denumerable, say A, then there is a bijection $f : \mathbb{N} \to A$. Also, since B is countable, there is a surjection $g : \mathbb{N} \to B$ (which could potentially be a bijection if B is denumerable, but may only be a surjection if B is finite). Define a map $h : \mathbb{N} \to A \cup B$ by $h(2n) = f(n)$ and $h(2n-1) = g(n)$ (i.e., send the n^{th} even number to $f(n)$ and the n^{th} odd number to $g(n)$). By construction, every element of A and every element of B appears in the image of h, so h is a surjection. Hence, $A \cup B$ is countable.

(ii) For a finite collection $A_1, \ldots, A_k$ of countable sets, we can prove by induction on k. The case $k = 2$ is part (i). Assume the union of k countable sets is countable. Then $A_1 \cup \cdots \cup A_{k+1} = (A_1 \cup \cdots \cup A_k) \cup A_{k+1}$, which is a union of two countable sets (by the induction hypothesis and the assumption on A_{k+1}), hence countable by part (i). Thus, the union of finitely many countable sets is countable.

7.11. Let A and B be denumerable sets. Prove that the set $A \times B$ is denumerable.

Since A and B are denumerable, there exist bijections $f : \mathbb{N} \to A$ and $g : \mathbb{N} \to B$. Consider the map $\Phi : \mathbb{N} \times \mathbb{N} \to A \times B$ defined by $\Phi(m,n) = (f(m), g(n))$. As f and g are bijective, so too is Φ. Thus, if we show that $\mathbb{N} \times \mathbb{N}$ is denumerable, it will follow that $A \times B$ is denumerable. To see this is the case, either appeal to the result you proved in Problem 7.7 and the fact that $\mathbb{N} \equiv \mathbb{Q}$ or, more simply, just arrange the elements of $\mathbb{N} \times \mathbb{N}$ in an infinite grid and traverse along diagonals:

$$(1,1),\ (1,2),(2,1),\ (1,3),(2,2),(3,1),\ \ldots$$

This traversal gives an explicit listing of all pairs in $\mathbb{N} \times \mathbb{N}$ in a single sequence. By mapping each $n \in \mathbb{N}$ to the corresponding term in the sequence, we produce a bijection $\mathbb{N} \to \mathbb{N} \times \mathbb{N}$. Therefore $\mathbb{N} \times \mathbb{N}$ and $A \times B$ are both denumerable.

7.12. Let A, B, and C be sets. Define $A \times B \times C = \{(a,b,c) \mid a \in A \text{ and } b \in B \text{ and } c \in C\}$. Prove that $A \times B \times C \equiv (A \times B) \times C$.

Define $\varphi : A \times B \times C \to (A \times B) \times C$ by $\varphi(a,b,c) = ((a,b),c)$ and let $\psi : (A \times B) \times C \to A \times B \times C$ be the function $\psi((a,b),c) = (a,b,c)$. Then, for all $(a,b,c) \in A \times B \times C$, we have $\psi(\varphi(a,b,c)) = \psi((a,b),c) = (a,b,c)$. Likewise, for all $((a,b),c) \in (A \times B) \times C$, we have $\varphi(\psi((a,b),c)) = \varphi(a,b,c) = ((a,b),c)$. Thus, $\psi = \varphi^{-1}$ and φ is a bijection (since it is invertible). Hence, $A \times B \times C \equiv (A \times B) \times C$.

7.13. For each natural number n, let $\mathbb{N}^n$ denote the set of ordered n-tuples of natural numbers, so $\mathbb{N}^3 = \mathbb{N} \times \mathbb{N} \times \mathbb{N}$, etc. Prove that $\mathbb{N}^n$ is countable.

We proceed by induction. For the base case of $n = 1$, observe that $\mathbb{N}^1 = \mathbb{N}$, which is denumerable (since $\mathbb{N} \equiv \mathbb{N}$) and hence countable. For the inductive step, let us assume $\mathbb{N}^n$ is countable. We will show $\mathbb{N}^{n+1}$ is countable as well. Since $\mathbb{N}^{n+1} \equiv \mathbb{N}^n \times \mathbb{N}$ by Exercise 7.12, we apply the inductive hypothesis and note that since $\mathbb{N}^n$ is countable and $\mathbb{N}$ is denumerable, $\mathbb{N}^{n+1}$ is equinumerous with the product of two countable sets (which are, in fact, both denumerable, as neither is finite). By Exercise 7.11, the product of two denumerable sets is denumerable, so $\mathbb{N}^{n+1}$ is as well and is therefore countable. By induction, $\mathbb{N}^n$ is countable for every $n \in \mathbb{N}$.

7.14. Let $\mathbb{S}$ denote the set of sequences of 0's and 1's, so a typical element of $\mathbb{S}$ looks like $(x_1, x_2, x_3, \ldots)$ where each x_i is either 0 or 1. Prove that $\mathbb{S}$ is uncountable.

For an element $s \in \mathbb{S}$, write s_k to denote the k^{th} term in s (since the elements of $\mathbb{S}$ are sequences of zeros and ones); i.e., $s = (s_1, s_2, \ldots)$. Suppose $f : \mathbb{N} \to \mathbb{S}$ is a function. We show that f cannot be injective by producing a new element $t \in S$ such that t is not in the image of $\mathbb{N}$ under f (i.e., $t \notin f(\mathbb{N}) = \{f(n) \in S | n \in \mathbb{N}\}$). To do this, we take

$$t_n = \begin{cases} 0, & \text{if } \left(f(n)\right)_n = 1, \\ 1, & \text{if } \left(f(n)\right)_n = 0. \end{cases}$$

That is, we let t be such that its n^{th} term differs from the n^{th} term of $f(n)$. By construction, t is in $\mathbb{S}$ (since it is a sequence of 0's and 1's). However, $t \neq f(n)$ for any $n \in \mathbb{N}$, because $t_n \neq \left(f(n)\right)_n$. Therefore, there is no surjection $f : \mathbb{N} \to \S$, which implies that $\mathbb{S}$ is not countable (that is, it is *uncountable*).