HOMEWORK 6

CHAPTER 6: THE REAL NUMBERS

6.1. Use the field axioms to show that $-0 = 0$ and $1^{-1} = 1$.

6.2. Prove that $(-a)b = -(ab) = a(-b)$ for all real numbers a and b.

6.3. Show that for any $a \in \mathbb{R}$, $(-a)(-a) = a^2$.

6.4. If $a > b \geq 0$ and $c > d \geq 0$, prove that $ac > bd$.

6.5. If $0 < a < b$, prove that $b^{-1} < a^{-1}$.

6.6. Let a and b be real numbers. Prove that if $a < 0$ and $b < 0$, then $ab > 0$.

6.7. Let a and b be real numbers. Prove that if $a > 0$ and $b < 0$, then $ab < 0$.

6.8. Let $a \in \mathbb{R}$. Prove that if $a < 0$, then $a^{-1} < 0$.

6.9. Let A be a nonempty subset of $\mathbb{R}$. Prove that if u and v are least upper bounds for A, then $u = v$.

6.10. Prove that if u is an upper bound for A and $u \in A$, then u is the least upper bound for A.

6.11. Show that every nonempty subset of $\mathbb{R}$ with a lower bound has a greatest lower bound.

6.12. Let A and B be two nonempty subsets of $\mathbb{R}$ such that $A \cup B = \mathbb{R}$. If A and B satisfy the further property that $a < b$ for every $a \in A$ and $b \in B$, then A and B form a *Dedekind cut* of $\mathbb{R}$. The Completeness Axiom is sometimes replaced with Dedekind's Axiom, which says that given any Dedekind cut of $\mathbb{R}$, either A has a largest element or B has a smallest element. Assuming the field and order axioms for $\mathbb{R}$, as well as their consequences, prove the following:

(i) The Completeness Axiom implies Dedekind's Axiom.

(ii) Dedekind's Axiom implies the Completeness Axiom.