# HOMEWORK 6

# CHAPTER 6: THE REAL NUMBERS

**6.1.** Use the field axioms to show that $-0 = 0$ and $1^{-1} = 1$.

First, we show that $-0 = 0$. By the definition of an additive inverse, $-0$ is the element of $\mathbb{R}$ such that $0 + (-0) = 0$. By the definition of the additive identity, $0 + a = a$ for all $a \in \mathbb{R}$. So, taking $a = -0$, we get $0 + (-0) = -0$. Combining these two observations, we produce our desired result:

$$-0 = 0 + (-0) = 0.$$

Now, we show that $1^{-1} = 1$. By the definition of a multiplicative inverse, $1^{-1}$ is the element of $\mathbb{R}$ such that $(1)(1^{-1}) = 1$. By the definition of the multiplicative identity, $1a = a$ for all $a \in \mathbb{R}$. Taking $a = 1^{-1}$, this yields $(1)(1^{-1}) = 1^{-1}$. Combining these two observations, we produce our desired result:

$$1^{-1} = (1)(1^{-1}) = 1.$$

**6.2.** Prove that $(-a)b = -(ab) = a(-b)$ for all real numbers $a$ and $b$.

First, we show that $(-a)b = -(ab)$ for all real numbers $a$ and $b$. To do so, we use item $(vi)$ of Theorem 6.2 from the lecture notes, which states that $-a = (-1)a$. Using this, we get the following via the associativity of multiplication:

$$\begin{aligned} (-a)b &= ((-1)a)b \quad \text{(Theorem 6.2 (vi))} \\ &= (-1)(ab) \quad \text{(Associativity of Multiplication)} \\ &= -(ab) \quad \text{(Theorem 6.2 (vi)).} \end{aligned}$$

Now, we show that $(-a)b = a(-b)$. Again, we use item $(vi)$ of Theorem 6.2 from the lecture notes, which states that $-a = (-1)a$. Using this, we get the following via the commutativity and associativity of multiplication:

$$
\begin{aligned}
(-a)b &= \big((-1)a\big)b \quad \text{(Theorem 6.2 (vi))}\\
&= \big(a(-1)\big)b \quad \text{(Commutativity of Multiplication)}\\
&= (a)\big((-1)b\big) \quad \text{(Associativity of Multiplication)}\\
&= a(-b) \quad \text{(Theorem 6.2 (vi)).}
\end{aligned}
$$

By the transitivity of equality, we may conclude that $(-a)b = -(ab) = a(-b)$ for all real numbers $a$ and $b$.

**Note**: There are certainly many other ways to prove this fact without using Theorem 6.2! If you are not sure whether your proof is correct, please feel free to reach out and ask me to check!

**6.3.** Show that for any $a \in \mathbb{R}$, $(-a)(-a) = a^2$.

We will show that for any $a \in \mathbb{R}$, $(-a)(-a) = a^2$. To do so, we use item $(vi)$ of Theorem 6.2 from the lecture notes, which states that $-a = (-1)a$, together with item $(ii)$ of Theorem 6.2 from the lecture notes, which states that $-(-a) = a$. To wit, observe that the commutativity and associativity of multiplication yields:

$$
\begin{aligned}
(-a)(-a) &= \big((-1)a\big)\big((-1)a\big) \quad \text{(Theorem 6.2 (vi))}\\
&= (-1)\big(a(-1)\big)a \quad \text{(Associativity of Multiplication)}\\
&= (-1)\big((-1)a\big)a \quad \text{(Commutativity of Multiplication)}\\
&= \big((-1)(-1)\big)(a^2) \quad \text{(Associativity of Multiplication)}\\
&= \big(-(-1)\big)(a^2) \quad \text{(Theorem 6.2 (vi))}\\
&= (1)(a^2) \quad \text{(Theorem 6.2 (ii))}\\
&= a^2 \quad \text{(Definition of Multiplicative Identity)}
\end{aligned}
$$

Therefore, $(-a)(-a) = a^2$ for any $a \in \mathbb{R}$.

**6.4.** If $a > b \geq 0$ and $c > d \geq 0$, prove that $ac > bd$.

Recall that the fourth order axiom states that if $a < b$ and $c > 0$, then $ac < bc$. Since $a > b \geq 0$ and $c > d \geq 0$, we have $ac > bc$ by the fourth order axiom (i.e. $a > b$ and $c > 0$, so $ac < bc$). Likewise, $bc > bd$ by the fourth order axiom (i.e. $c > d$ and $b > 0$, so $bc > bd$). Via transitivity, this yields $ac > bd$.

**6.5.** If $0 < a < b$, prove that $b^{-1} < a^{-1}$.

We have $0 < a < b$ and, in particular, $a, b > 0$, so $a^{-1}$ and $b^{-1}$ exist and are positive (by part (i) of Theorem 6.5 in the lecture notes). Accordingly, we may multiply the inequality $0 < a < b$ by the positive number $a^{-1}b^{-1} > 0$. Since multiplication by a positive number preserves inequalities (this is part (i) of Theorem 6.4 in the lecture notes), we get

$$0 \cdot a^{-1}b^{-1} < a \cdot a^{-1}b^{-1} < b \cdot a^{-1}b^{-1}.$$

This simplifies (using the commutativity of multiplication) to

$$0 < b^{-1} < a^{-1}.$$

Therefore, we may conclude that $b^{-1} < a^{-1}$ whenever $0 < a < b$.

**6.6.** Let $a$ and $b$ be real numbers. Prove that if $a < 0$ and $b < 0$, then $ab > 0$.

If $a < 0$, then $-a > 0$ (this is part (i) of Theorem 6.3 in the lecture notes in combination with part (ii) of Theorem 6.2). Likewise, if $b < 0$, then $-b > 0$ (again, for the same reason). Thus $-a$ and $-b$ are both positive real numbers, and the product of two positive numbers is positive (this is part (i) of Theorem 6.4, which is a direct consequence of the fourth order axiom), so we may conclude that:

$$(-a)(-b) > 0.$$

But $(-a)(-b) = ab$ (via two applications of part (vi) of Theorem 6.2, the associativity of multiplication, and part (ii) of Theorem 6.2). Therefore, we may conclude that $ab > 0$.

**6.7.** Let $a$ and $b$ be real numbers. Prove that if $a > 0$ and $b < 0$, then $ab < 0$.

If $b < 0$, then $-b > 0$ (this is part (i) of Theorem 6.3). Since $a > 0$ as well, the product $a(-b)$ is positive (this is part (i) of Theorem 6.4, which is a direct consequence of the fourth order axiom); i.e., $a(-b) > 0$.

But $a(-b) = -(ab)$ (via Problem 6.2 in this assignment), so we have $-(ab) > 0$. This is equivalent to $ab < 0$ (by part (i) of Theorem 6.3 in the lecture notes in combination with part (ii) of Theorem 6.2).

**6.8.** Let $a \in \mathbb{R}$. Prove that if $a < 0$, then $a^{-1} < 0$.

Since $a < 0$, we have $-a > 0$ (this is part (i) of Theorem 6.3 in the lecture notes in combination with part (ii) of Theorem 6.2). Because $-a > 0$, its inverse $(-a)^{-1}$ exists and is positive (by part (i) of Theorem 6.5 in the lecture notes). That is, $(-a)^{-1} > 0$.

Now, recall that part (iii) of Theorem 6.2 states that if $a \neq 0$ and $ab = ac$, then $b = c$. Accordingly, we have

$$\begin{aligned}
-(a^{-1})(-a) &= (-1)(a^{-1})(-1)a \quad \text{(Theorem 6.2 (vi))} \\
&= (-1)(-1)a^{-1}a \quad \text{(Commutativity of Multiplication)} \\
&= (-1)(-1) \quad \text{(Definition of Multiplicative Inverse)} \\
&= -(-1) \quad \text{(Theorem 6.2 (vi))} \\
&= 1 \quad \text{(Theorem 6.2 (ii))} \\
&= (-a)^{-1}(-a) \quad \text{(Definition of Multiplicative Inverse)}
\end{aligned}$$

Hence, $-(a^{-1})(-a) = (-a^{-1})(-a)$, so we may conclude $-(a^{-1}) = (-a)^{-1}$.

Thus, as $(-a)^{-1} > 0$, we may conclude that $-(a^{-1}) > 0$ as well, which implies $a^{-1} < 0$ (by part (i) of Theorem 6.3 in the lecture notes in combination with part (ii) of Theorem 6.2).

**6.9.** Let $A$ be a nonempty subset of $\mathbb{R}$. Prove that if $u$ and $v$ are least upper bounds for $A$, then $u = v$.

Since $u$ is a least upper bound (supremum) of $A$, it is an upper bound for $A$, and moreover it is less than or equal to every upper bound of $A$. In particular, $v$ is an upper bound of $A$, so

$$u \leq v.$$

Similarly, since $v$ is a least upper bound of $A$, it is less than or equal to every upper bound of $A$. In particular, $u$ is an upper bound of $A$, so

$$v \leq u.$$

From $u \leq v$ and $v \leq u$, the antisymmetry of the order on $\mathbb{R}$ implies $u = v$. Hence the least upper bound is unique.

**6.10.** Prove that if $u$ is an upper bound for $A$ and $u \in A$, then $u$ is the least upper bound for $A$.

Suppose $u$ is an upper bound for $A$ and that $u \in A$. Since $u$ is an upper bound for $A$, every $a \in A$ satisfies $a \leq u$.

Now, let $v$ be any upper bound for $A$. Because $u \in A$ and $v$ is an upper bound, we have $u \leq v$. This shows that $u$ is less than or equal to every upper bound of $A$. Thus $u$ is an upper bound of $A$ that is less than or equal to every upper bound of $A$, which is exactly the definition of the least upper bound. Therefore $u = \sup A$.

**6.11.** Show that every nonempty subset of $\mathbb{R}$ with a lower bound has a greatest lower bound.

Let $S \subset \mathbb{R}$ be nonempty and suppose $S$ has a lower bound. Consider the set

$$-S := \{-s \mid s \in S\}.$$

Then $-S$ is nonempty. If $m$ is a lower bound for $S$, i.e. $m \leq s$ for all $s \in S$, then $-m$ is an upper bound for $-S$ because for each $s \in S$ we have $-s \leq -m$. Thus $-S$ is a nonempty subset of $\mathbb{R}$ that is bounded above. By the Completeness Axiom, $-S$ has a least upper bound; denote it by $u = \sup(-S)$.

We claim that $-u$ is the greatest lower bound (infimum) of $S$. To do so, we first show $-u$ is a lower bound of $S$. Let $s \in S$. Then $-s \in -S$. Since $u$ is an upper bound for $-S$, we have $-s \leq u$, which implies $s \geq -u$. Thus $-u \leq s$ for all $s \in S$, so $-u$ is a lower bound. Now, let $l$ be any lower bound for $S$. Then $l \leq s$ for all $s \in S$, so for all $s \in S$ we have $-s \leq -l$. Hence $-l$ is an upper bound for $-S$. Since $u$ is the least upper bound of $-S$, we have $u \leq -l$, which implies $-u \geq l$. Accordingly, every lower bound $l$ of $S$ satisfies $l \leq -u$, meaning $-u$ is greater than or equal to any other lower bound. Thus $-u = \inf S$ is the greatest lower bound of $S$.

**6.12.** Let $A$ and $B$ be two nonempty subsets of $\mathbb{R}$ such that $A \cup B = \mathbb{R}$. If $A$ and $B$ satisfy the further property that $a < b$ for every $a \in A$ and $b \in B$, then $A$ and $B$ form a *Dedekind cut* of $\mathbb{R}$. The Completeness Axiom is sometimes replaced with Dedekind's Axiom, which says that given any Dedekind cut of $\mathbb{R}$, either $A$ has a largest element or $B$ has a smallest element. Assuming the field and order axioms for $\mathbb{R}$, as well as their consequences, prove the following:

(i) The Completeness Axiom implies Dedekind's Axiom.

(ii) Dedekind's Axiom implies the Completeness Axiom.

(i) Assume the Completeness Axiom: every nonempty subset of $\mathbb{R}$ that is bounded above has a least upper bound. Let $(A, B)$ be a Dedekind cut. Then $A$ is nonempty and bounded above by any element of $B$. By completeness, $A$ has a least upper bound; call it $u = \sup A$. We

claim that either $u \in A$ (so $A$ has a largest element) or $u \in B$ (so $B$ has a smallest element). Since $A \cup B = \mathbb{R}$ and $A, B$ are disjoint, $u$ must lie in exactly one of $A$ or $B$.
First, suppose $u \in A$. We show $u$ is the largest element of $A$. Suppose toward a contradiction that there exists $a \in A$ with $a > u$. Then $u$ would not be an upper bound of $A$, contradicting $u = \sup A$. Thus no element of $A$ is greater than $u$, and $u$ is the largest element of $A$.
Now, suppose $u \in B$. We show $u$ is the smallest element of $B$. Suppose there exists $b \in B$ with $b < u$. Because $u$ is the least upper bound of $A$ and $b < u$, $b$ cannot be an upper bound of $A$. Hence there exists $a \in A$ such that $a > b$. But then $a \in A$ and $b \in B$ with $a > b$, contradicting the defining property of the Dedekind cut that $a < b$ for all $a \in A$, $b \in B$. Thus no element of $B$ can be less than $u$, so $u$ is the smallest element of $B$.
Hence, given a Dedekind cut, either $A$ has a largest element or $B$ has a smallest element and Dedekind's Axiom holds.

(ii) Assume Dedekind's Axiom: every Dedekind cut $(A, B)$ satisfies that either $A$ has a largest element or $B$ has a smallest element. Let $S \subset \mathbb{R}$ be nonempty and bounded above. We will show that $S$ has a least upper bound. To do so, define

$$B = \{x \in \mathbb{R} : x \text{ is an upper bound for } S\},$$

and let $A = \mathbb{R} \setminus B$. Then $B$ is nonempty (since $S$ is bounded above) and $A$ is nonempty (because any $s \in S$ is not an upper bound for $S$, so $s \in A$). Also $A \cup B = \mathbb{R}$ by construction, and $A \cap B = \emptyset$. First, we show $(A, B)$ is a Dedekind cut; i.e., we show that $a < b$ for every $a \in A$ and $b \in B$. Suppose, to the contrary, that there exist $a \in A$ and $b \in B$ with $a \geq b$. Since $b$ is an upper bound for $S$ and $a \geq b$, $a$ is also an upper bound for $S$. Thus $a \in B$, contradicting $a \in A$. Hence $a < b$ for all $a \in A$, $b \in B$, so $(A, B)$ is indeed a Dedekind cut.

By Dedekind's Axiom, either $A$ has a largest element or $B$ has a smallest element. We now show that $A$ cannot have a largest element, therefore implying that $B$ has a smallest element (which we will show is the least upper bound of $S$). Assume for contradiction that $a_0$ is the largest element of $A$. Since $a_0 \in A$, it is not an upper bound for $S$. Thus there exists $s \in S$ with $s > a_0$. But any $s \in S$ is not an upper bound for $S$, so $s \in A$. This contradicts that $a_0$ is the largest element of $A$. Therefore $A$ has no largest element. Thus, by Dedekind's Axiom, $B$ must have a smallest element; call it $u$. We show that $u$ is the least upper bound of $S$.
First, we note that $u$ is an upper bound for $S$ by the definition of $B$. To see that $u$ is the least such upper bound, let $v < u$. Then $v \notin B$ since $u$ is the smallest element of $B$. Hence $v \in A$, which means $v$ is not an upper bound for $S$. Thus there exists $s \in S$ with $s > v$. So no real number less than $u$ is an upper bound for $S$.
Hence $u$ is an upper bound of $S$ and every smaller real fails to be an upper bound. Therefore $u = \sup S$ exists, and the Completeness Axiom holds.