HOMEWORK 5

CHAPTER 5: FUNCTIONS

5.1. Let A be a nonempty set. Explain why there are no functions from A to $\emptyset$.

5.2. Is standard division a binary operation on $\mathbb{N}$? on $\mathbb{R}$? Justify your answer in each case.

5.3. Is the dot product a binary operation on $\mathbb{R}^3$? Recall that the dot product is defined by $(a,b,c)\cdot(d,e,f) = ad + be + cf$.

5.4. Determine whether or not each of the following binary operations on $\mathbb{R}$ is (a) commutative, (b) associative.

(i) $a * b = |a - b|$.

(ii) $a * b = \frac{a}{b^2+1}$.

(iii) $a * b = \max\{a, b\}$.

5.5. Let $f : X \to Y$ be a function; let A and B be subsets of X. For $Z \subset X$ define $f(Z) = \{f(z) \mid z \in Z\}$. Determine which of the following are true. If a statement is true, prove it. If a statement is false, find a counterexample.

(i) $f(A \cup B) \subset f(A) \cup f(B)$

(ii) $f(A \cup B) \supset f(A) \cup f(B)$

(iii) $f(A \cap B) \subset f(A) \cap f(B)$

(iv) $f(A \cap B) \supset f(A) \cap f(B)$

(v) $f(X \setminus A) \subset Y \setminus f(A)$

(vi) $f(X \setminus A) \supset Y \setminus f(A)$

5.6. For each of the false statements in the previous exercise, determine whether or not they are true under the following conditions. Prove or give a counterexample in each case.

(i) $f : X \to Y$ is an injective function.

(ii) $f : X \to Y$ is a surjective function.

5.7. Let $A = \{1, 2, 3\}$. For each of the following, either find a function satisfying the indicated properties or prove that no such function exists.

(i) A bijective function $f : A \to A$ other than the identity function.

(ii) An injective function $g : A \to A$ that is not surjective.

(iii) A surjective function $h : A \to A$ that is not injective.

(iv) A function $j : A \to A$ that is neither injective nor surjective.

5.8. Let $A = \{1, 2, 3\}$ and $B = \{4, 5\}$. For each of the following, either find a function satisfying the indicated properties or prove that no such function exists.

(i) A bijective function $f : A \to B$.

(ii) An injective function $g : A \to B$ that is not surjective.

(iii) A surjective function $h : A \to B$ that is not injective.

(iv) A function $j : A \to B$ that is neither injective nor surjective.

5.9. For each of the following, either find a function satisfying the indicated properties or prove that no such function exists.

(i) A bijective function $f : \mathbb{N} \to \mathbb{N}$ other than the identity.

(ii) An injective function $g : \mathbb{N} \to \mathbb{N}$ that is not surjective.

(iii) A surjective function $h : \mathbb{N} \to \mathbb{N}$ that is not injective.

(iv) A function $j : \mathbb{N} \to \mathbb{N}$ that is neither injective nor surjective.

5.10. Let A be a nonempty set and $\sim$ an equivalence relation on A; let $\widehat{A}$ be the set of all equivalence classes. Prove that there is an injective function $f : \widehat{A} \to A$.

5.11. Find examples of sets X, Y, and Z and functions $f : X \to Y$ and $g : Y \to Z$ so that $g \circ f : X \to Z$ is surjective but $f : X \to Y$ is not surjective.

5.12. The standard integer addition operation $+$ is a function from $\mathbb{Z}^2 \to \mathbb{Z}$. Show that if $a \equiv_5 b$ and $c \equiv_5 d$, then $a + c \equiv_5 b + d$. This shows that addition is *well-defined* with respect to this equivalence relation.

5.13. Let $f : X \to Y$ be an invertible function and let $f^{-1} : Y \to X$ denote the inverse of f. Show that f^{-1} is bijective and that f is the inverse of f^{-1}.

5.14. Let $f : X \to Y$ be a function and suppose that the function $g : Y \to X$ is the inverse of f. Show that the composition $g \circ f : X \to X$ is the identity function, i.e. that $g(f(x)) = x$ for every $x \in X$.