HOMEWORK 5

CHAPTER 5: FUNCTIONS

5.1. Let A be a nonempty set. Explain why there are no functions from A to $\emptyset$.

There are no functions from A to the empty set because there are no there are no relations from A to the empty set and a function is a particular kind of relation. That is, a relation from A to the empty set is simply a subset of $A \times \emptyset$, but observe that $A \times \emptyset = \{(x, y) \mid x \in A \wedge y \in \emptyset\} = \emptyset$ (since there are no elements y such that $y \in \emptyset$).

5.2. Is standard division a binary operation on $\mathbb{N}$? On $\mathbb{R}$? Justify your answer in each case.

Standard division is not a binary operation on $\mathbb{N}$, nor is it a binary operation on $\mathbb{R}$. To see that it is not a binary operation on $\mathbb{N}$, observe, for instance, that $1 \div 2 \notin \mathbb{N}$. Likewise, to see that it is not a binary operation on $\mathbb{R}$, observe that $2 \div 0 \notin \mathbb{R}$ (indeed, it is not even defined!).

5.3. Is the dot product a binary operation on $\mathbb{R}^3$? Recall that the dot product is defined by $(a, b, c) \cdot (d, e, f) = ad + be + cf$.

No, the dot product is not a binary operation on $\mathbb{R}^3$. While the dot product is well-defined for all pairs of vectors in $\mathbb{R}^3$, it is not the case that the result of performing the dot product on two vectors in $\mathbb{R}^3$ is, itself, a vector. That is, $(a, b, c) \cdot (d, e, f) \notin \mathbb{R}^3$ (in fact, $(a, b, c) \cdot (d, e, f) \in \mathbb{R}$).

5.4. Determine whether or not each of the following binary operations on $\mathbb{R}$ is (a) commutative, (b) associative.

(i) $a * b = |a - b|$.

(ii) $a * b = \frac{a}{b^2+1}$.

(iii) $a * b = \max\{a, b\}$.

(i) The binary operation $a * b = |a - b|$ is commutative, but is not associative. To see commutativity, observe that

$$b * a = |b - a| = |(-1)(a - b)| = |-1|\ |a - b| = |a - b| = a * b.$$

Likewise, to see that it is not associative, observe the following counterexample to the claim that it is:

$$1 * (1 * 2) = 1 * |1 - 2| = 1 * 1 = 0 \neq 2 = 0 * 2 = |1 - 1| * 2 = (1 * 1) * 2.$$

(ii) The binary operation $a * b = \frac{a}{b^2+1}$ is neither commutative nor associative. To see that it is not commutative, observe the following counterexample to the claim that it is:

$$1 * 2 = \frac{1}{2^2+1} = \frac{1}{5} \neq 1 = \frac{2}{2} = \frac{2}{1^2+1} = 2 * 1.$$

Likewise, to see that it is not associative, observe the following counterexample to the claim that it is:

$$1 * (2 * 1) = 1 * 1 = \frac{1}{2} \neq \frac{1}{10} = \frac{\frac{1}{5}}{1^2+1} = \frac{1}{5} * 1 = (1 * 2) * 1.$$

(iii) The binary operation $a * b = \max\{a, b\}$ is both commutative and associative. To begin, first note that

$$\max\{a, b\} = \begin{cases} a & a \geq b \\ b & b \geq a \end{cases}$$

Using this, we can see that this operation is commutative by observing the following:

$$a * b = \max\{a, b\} = \begin{cases} a & a \geq b \\ b & b \geq a \end{cases} = \begin{cases} b & b \geq a \\ a & a \geq b \end{cases} = \max\{b, a\} = b * a.$$

Likewise, to see that it is associative, observe the following:

$$a * (b * c) = \max\{a, \max\{b, c\}\} = \begin{cases} a & a \geq \max\{b, c\} \\ \max\{b, c\} & \max\{b, c\} \geq a \end{cases} = \begin{cases} a & a \geq b \geq c \\ a & a \geq c \geq b \\ b & b \geq a \geq c \\ b & b \geq c \geq a \\ c & c \geq a \geq b \\ c & c \geq b \geq a \end{cases}$$

$$(a * b) * c = \max\{\max\{a, b\}, c\} = \begin{cases} \max\{a, b\} & \max\{a, b\} \geq c \\ c & c \geq \max\{a, b\} \end{cases} = \begin{cases} a & a \geq b \geq c \\ a & a \geq c \geq b \\ b & b \geq a \geq c \\ b & b \geq c \geq a \\ c & c \geq a \geq b \\ c & c \geq b \geq a \end{cases}$$

5.5. Let $f : X \to Y$ be a function; let A and B be subsets of X. For $Z \subset X$ define $f(Z) = \{f(z) \mid z \in Z\}$. Determine which of the following are true. If a statement is true, prove it. If a statement is false, find a counterexample.

(i) $f(A \cup B) \subset f(A) \cup f(B)$

(ii) $f(A \cup B) \supset f(A) \cup f(B)$

(iii) $f(A \cap B) \subset f(A) \cap f(B)$

(iv) $f(A \cap B) \supset f(A) \cap f(B)$

(v) $f(X \setminus A) \subset Y \setminus f(A)$

(vi) $f(X \setminus A) \supset Y \setminus f(A)$

(i) The statement $f(A \cup B) \subset f(A) \cup f(B)$ is true. To see this, let $y \in f(A \cup B)$. Then, $y = f(x)$ for some $x \in A \cup B$. Thus, $x \in A$ or $x \in B$, so $y \in f(A) \cup f(B)$. As y was arbitrary, we may conclude $f(A \cup B) \subset f(A) \cup f(B)$.

(ii) The statement $f(A \cup B) \supset f(A) \cup f(B)$ is true. To see this, let $y \in f(A) \cup f(B)$. Then, $y = f(a)$ for some $a \in A$ or $y = f(b)$ for some $b \in B$. In either case, a or b both belong to $A \cup B$, so $y \in f(A \cup B)$. As y was arbitrary, we may conclude $f(A \cup B) \supset f(A) \cup f(B)$.

(iii) The statement $f(A \cap B) \subset f(A) \cap f(B)$ is true. To see this, let $y \in f(A \cap B)$. Then, $y = f(x)$ for some $x \in A \cap B$. This implies that $x \in A$ and $x \in B$, so $y \in f(A)$ and $y \in f(B)$, which implies that $y \in f(A) \cap f(B)$. As y was arbitrary, we may conclude $f(A \cap B) \subset f(A) \cap f(B)$.

(iv) The statement $f(A \cap B) \supset f(A) \cap f(B)$ is false. For a counterexample, let $X = \{1,2\}$, $Y = \{a,b\}$, $A = \{1\} \subseteq X$, $B = \{2\} \subseteq X$, and let f be the constant function $f(x) = a$. Then, $A \cap B = \emptyset$, so $f(A \cap B) = \emptyset$, while $f(A) \cap f(B) = \{a\} \cap \{a\} = \{a\}$. Clearly, $f(A) \cap f(b) = \{a\} \not\subseteq \emptyset = f(A \cap B)$.

(v) The statement $f(X \setminus A) \subset Y \setminus f(A)$ is false. For a counterexample, let $X = \{1,2\}$, $Y = \{a,b\}$, $A = \{1\} \subseteq X$, and let f be the constant function $f(x) = a$. Then, $X \setminus A = \{2\}$, so $f(X \setminus A) = f(\{2\}) = \{a\}$, while $Y \setminus f(A) = \{a,b\} \setminus \{a\} = \{b\}$. Clearly, $f(X \setminus A) = \{a\} \not\subseteq Y \setminus f(A) = \{b\}$.

(vi) The statement $f(X \setminus A) \supset Y \setminus f(A)$ is false. For a counterexample, let $X = \{1,2\}$, $Y = \{a,b\}$, $A = \{1\} \subseteq X$, and let f be the constant function $f(x) = a$. Then, $X \setminus A = \{2\}$, so $f(X \setminus A) = f(\{2\}) = \{a\}$, while $Y \setminus f(A) = \{a,b\} \setminus \{a\} = \{b\}$. Clearly, $f(X \setminus A) = \{a\} \not\supseteq Y \setminus f(A) = \{b\}$.

5.6. For each of the false statements in the previous exercise, determine whether or not they are true under the following conditions. Prove or give a counterexample in each case.

(i) $f : X \to Y$ is an injective function.

(ii) $f : X \to Y$ is a surjective function.

In the previous exercise, items (iv), (v) and (vi) were shown to be false by counterexample. We now consider them under the following additional assumptions:

(i) Suppose $f : X \to Y$ is an injective function.

(iv) The statement $f(A \cap B) \supset f(A) \cap f(B)$ is true. To see this, let $y \in f(A) \cap f(B)$. Then, there exists $a \in A$ such that $f(a) = y$ and there exists $b \in B$ such that $f(b) = y$. By injectivity, $a = b$. So, $a \in B$ and $b \in A$. That is, $y \in f(A \cap B)$. As y was arbitrary, we may conclude $f(A \cap B) \supset f(A) \cap f(B)$.

(v) The statement $f(X \setminus A) \subset Y \setminus f(A)$ is true. To see this, let $y \in f(X \setminus A)$. Then, $y = f(x)$ for some $x \in X \setminus A$. Hence, $x \notin A$. Now, if $y \in f(A)$, then $y = f(a)$ for some $a \in A$. By injectivity, this implies that $x = a$, contradicting $x \notin A$. So, $y \notin f(A)$. That is, $y \in Y \setminus f(A)$. As y was arbitrary, we may conclude $f(X \setminus A) \subset Y \setminus f(A)$.

(vi) The statement $f(X \setminus A) \supset Y \setminus f(A)$ is false even when f is injective. For a counterexample, let $X = \{1,2\}$, $Y = \{a,b,c\}$, $A = \{1\} \subseteq X$, $B = \{2\} \subseteq X$, and let f be the constant function

$$f(x) = \begin{cases} a & x = 1 \\ b & x = 2 \end{cases}.$$

Then, $X \setminus A = \{2\}$, so $f(X \setminus A) = f(\{2\}) = \{b\}$, while $Y \setminus f(A) = \{a,b,c\} \setminus \{a\} = \{b,c\}$. Clearly, $f(X \setminus A) = \{b\} \not\supseteq Y \setminus f(A) = \{b,c\}$.

(ii) Suppose $f : X \to Y$ is a surjective function.

(iv) The statement $f(A \cap B) \supset f(A) \cap f(B)$ is false, even when f is surjective. For a counterexample, let $X = \{1,2\}$, $Y = \{a\}$, $A = \{1\} \subseteq X$, $B = \{2\} \subseteq X$, and let f be the constant function $f(x) = a$ (which is indeed surjective, since $Y = \{a\}$). Then, $A \cap B = \emptyset$, so $f(A \cap B) = \emptyset$, while $f(A) \cap f(B) = \{a\} \cap \{a\} = \{a\}$. Clearly, $f(A) \cap f(b) = \{a\} \not\subseteq \emptyset = f(A \cap B)$.

(v) The statement $f(X \setminus A) \subset Y \setminus f(A)$ is false, even when f is surjective. For a counterexample, let $X = \{1,2\}$, $Y = \{a\}$, $A = \{1\} \subseteq X$, and let f be the constant function $f(x) = a$. Then, $X \setminus A = \{2\}$, so $f(X \setminus A) = f(\{2\}) = \{a\}$, while $Y \setminus f(A) = \{a\} \setminus \{a\} = \emptyset$. Clearly, $f(X \setminus A) = \{a\} \not\subseteq Y \setminus f(A) = \emptyset$.

(vi) The statement $f(X \setminus A) \supset Y \setminus f(A)$ is true if f is surjective. To see this, let $y \in Y \setminus f(A)$. Therefore, $y \notin f(A)$. Since f is surjective, there exists $x \in X$ such that $f(x) = Y$. If $x \in A$, then $y \in f(A)$, which contradicts our assumption that $y \notin f(A)$. So, $x \in X \setminus A$, and we may conclude that $y = f(x) \in f(X \setminus A)$. As y was arbitrary, $f(X \setminus A) \supset Y \setminus f(A)$.

5.7. Let $A = \{1,2,3\}$. For each of the following, either find a function satisfying the indicated properties or prove that no such function exists.

(i) A bijective function $f : A \to A$ other than the identity function.

(ii) An injective function $g : A \to A$ that is not surjective.

(iii) A surjective function $h : A \to A$ that is not injective.

(iv) A function $j : A \to A$ that is neither injective nor surjective.

Let $A = \{1,2,3\}$.

(i) An example of a bijective function $f : A \to A$ other than the identity function is the function $f = \{(1,3),(2,2),(3,1)\}$.

(ii) There does not exist an injective function $g : A \to A$ that is not surjective. To see this, observe that if g is injective, then $g(x) = g(y)$ implies $x = y$. Hence, $g(1) \neq g(2)$, $g(1) \neq g(3)$, and $g(2) \neq g(3)$. Accordingly, $g(A) = \{g(a) \mid a \in A\}$ must have at least 3 elements. But, since the codomain of g is A itself, this means $g(A) = A$ and g is necessarily surjective.

(iii) There does not exist a surjective function $h : A \to A$ that is not injective. To see this, observe that if h is surjective, then there exist values $a_1, a_2, a_3 \in A$ such that $h(a_1) = 1$, $h(a_2) = 2$, and $h(a_3) = 3$. Since h is a function, necessarily $a_1 \neq a_2$, $a_1 \neq a_3$, and $a_2 \neq a_3$. But this means that $\{a_1, a_2, a_3\} = A$, since A only has three distinct elements. So, there exists no values $x, y \in A$ with $x \neq y$ such that $h(x) = h(y)$. That is, h is necessarily injective.

(iv) It is straightforward to construct a function $j : A \to A$ that is neither injective nor surjective: Consider, for example, a constant function, such as $j(x) = 1$ for all $x \in A$. This function is not surjective, as $j(x) \neq 2$ for all $x \in A$, nor is it injective as $j(1) = 1 = j(2)$ and $1 \neq 2$.

5.8. Let $A = \{1,2,3\}$ and $B = \{4,5\}$. For each of the following, either find a function satisfying the indicated properties or prove that no such function exists.

(i) A bijective function $f : A \to B$.

(ii) An injective function $g : A \to B$ that is not surjective.

(iii) A surjective function $h : A \to B$ that is not injective.

(iv) A function $j : A \to B$ that is neither injective nor surjective.

Let $A = \{1,2,3\}$ and $B = \{4,5\}$.

(i) There does not exist an bijective function $f : A \to B$. To see this, we show that there does not exist an injective function $f : A \to B$. To see this, observe that if f is injective, then $f(x) = f(y)$ implies $x = y$. Hence, $f(1) \neq f(2)$, $f(1) \neq f(3)$, and $f(2) \neq f(3)$. Accordingly, $f(A) = \{f(a) \mid a \in A\}$ must have at least 3 elements. But, since the codomain of f is B and B has only two elements, no such function can exist. As a bijective function is necessarily also injective, this implies that no bijective function $f : A \to B$ exists.

(ii) There does not exist an injective function $g : A \to B$ that is not surjective due to item (i) as, in fact, there does not exist an injective function $g : A \to B$, regardless of its surjectivity or lack of surjectivity.

(iii) An example of a surjective function $h : A \to B$ that is not injective is the function

$$h(x) = \begin{cases} 4 & a \in \{1,2\} \\ 5 & a = 3 \end{cases}.$$

To see this function is surjective, observe that $f(1) = 4$ and $f(3) = 5$, so it is indeed the case that for all $b \in B$ there exists $a \in A$ such taht $h(a) = b$. To see that this function is not injective, observe that $h(1) = h(2)$ but $1 \neq 2$.

(iv) It is straightforward to construct a function $j : A \to B$ that is neither injective nor surjective: Consider, for example, a constant function, such as $j(x) = 4$ for all $x \in A$. This function is not surjective, as $j(x) \neq 5$ for all $x \in A$, nor is it injective as $j(1) = j(2)$ and $1 \neq 2$.

5.9. For each of the following, either find a function satisfying the indicated properties or prove that no such function exists.

(i) A bijective function $f : \mathbb{N} \to \mathbb{N}$ other than the identity.

(ii) An injective function $g : \mathbb{N} \to \mathbb{N}$ that is not surjective.

(iii) A surjective function $h : \mathbb{N} \to \mathbb{N}$ that is not injective.

(iv) A function $j : \mathbb{N} \to \mathbb{N}$ that is neither injective nor surjective.

(i) An example of a bijective function $f : \mathbb{N} \to \mathbb{N}$ other than the identity is the function

$$f(n) = \begin{cases} 2 & n = 1 \\ 1 & n = 2 \\ n & n \geq 3 \end{cases}$$

To see that this function is surjective, it is straightforward to see that for all $m \in \mathbb{N}$ there exists a value n such that $f(n) = m$: if $m = 2$, then $n = 1$; if $m = 1$, then $n = 2$; and if $m \geq 3$, $n = m$. Likewise, to see that this function is injective, observe that $f(1) \neq f(2)$, so if $f(n) = f(m)$, then $n, m \notin \{1,2\}$. So, if $f(n) = f(m)$ for $n, m \notin \{1,2\}$, observe that this implies, by the definition of f, that $n = m$.

(ii) An example of an injective function $g : \mathbb{N} \to \mathbb{N}$ that is not surjective is the function $g(n) = n + 1$. There exists no value n such that $g(n) = 1$, so g is certainly not surjective. However, if $g(n) = g(m)$, then $n + 1 = m + 1$ and, subtracting one from both sides of this equality, $n = m$. So, g is indeed injective.

(iii) An example of a surjective function $h : \mathbb{N} \to \mathbb{N}$ that is not injective is the the function

$$h(n) = \begin{cases} 1 & n = 1 \\ 1 & n = 2 \\ n - 1 & n \geq 3 \end{cases}.$$

To see that this function is surjective, observe that if $m = 1$, then $h(1) = 1$, while if $m \geq 2$, then $h(m+1) = (m+1) - 1 = m$. So, for all values $m \in \mathbb{N}$, there does indeed exist a value $n \in \mathbb{N}$ such that $h(n) = m$; i.e., h is indeed surjective.

(iv) An example of a function $j : \mathbb{N} \to \mathbb{N}$ that is neither injective nor surjective is the constant function $j(n) = 1$. There exists no value n such that $j(n) = 2$, so j is not surjective. Likewise, $j(1) = 1 = j(2)$, so j is not injective.

5.10. Let A be a nonempty set and $\sim$ an equivalence relation on A; let $\widehat{A}$ be the set of all equivalence classes. Prove that there is an injective function $f : \widehat{A} \to A$.

Let A be a nonempty set and $\sim$ an equivalence relation on A; let $\widehat{A}$ be the set of all equivalence classes. For each equivalence class $S \in \widehat{A}$, pick a single element $a_S \in A$ such that $[a_S] = S$, where $[x] = \{y \in A \mid x \sim y\}$. As no equivalence class is empty, this implies that that the function $f : \widehat{A} \to A$ defined by $f(S) = a_S$ is well-defined. As equivalence classes are either disjoint or identical (i.e. either $[a] = [b]$ or $[a] \cap [b] = \varnothing$ for all $a, b \in A$), observe that $f(S) = f(Q)$ implies $S = [f(S)] = [a_S] = [a_Q] = [f(Q)] = Q$. That is, f is injective.

Note: This proof actually requires a very special assumption: Namely, it requires that we assume it is possible to select an element from each equivalence class of A under $\sim$. The assumption that we can select an element from any collection of nonempty sets and form a new set from the result is known as the *axiom of choice* and is actually much more interesting than it first seems. I encourage you to go read the Wikipedia page about it sometime!

5.11. Find examples of sets X, Y, and Z and functions $f : X \to Y$ and $g : Y \to Z$ so that $g \circ f : X \to Z$ is surjective but $f : X \to Y$ is not surjective.

Let $X = \mathbb{R}$, $Y = \mathbb{R}$ and $Z = \{1\}$. Let f be the constant function $f(x) = 1$ and let g be the constant function $g(x) = 1$. Then $(g \circ f)(x) = g(f(x)) = 1$ for all x. So, $g \circ f : X \to Z$ is certainly surjective (since $Z = \{1\}$). However, $f : X \to Y$ is not surjective, since $Y = \mathbb{R}$ and f is constant.

5.12. The standard integer addition operation $+$ is a function from $\mathbb{Z}^2 \to \mathbb{Z}$. Show that if $a \equiv_5 b$ and $c \equiv_5 d$, then $a + c \equiv_5 b + d$. This shows that addition is *well-defined* with respect to this equivalence relation.

Suppose $a \equiv_5 b$ and $c \equiv_5 d$. By definition, this means that there exist integers n and m such that $a - b = 5n$ and $c - d = 5m$. Accordingly, $a + c = (5n + b) + (5m + d) = 5(n + m) + b + d$. That is, $(a + c) - (b + d) = 5(n + m)$ and $a + c \equiv_5 b + d$.

5.13. Let $f : X \to Y$ be an invertible function and let $f^{-1} : Y \to X$ denote the inverse of f. Show that f^{-1} is bijective and that f is the inverse of f^{-1}.

Let $f : X \to Y$ be an invertible function and let $f^{-1} : Y \to X$ denote the inverse of f. Since f is invertible, f^{-1} is a function (and not merely a relation). Accordingly, we will show that f^{-1} is bijective by showing that it is invertible, since we have shown in lecture that a function is invertible if and only if it is bijective. By definition, f^{-1} is invertible if its inverse (i.e. the set $\{(x, y) \mid (y, x) \in f^{-1}\}$) is a function and not merely a relation. So, observe that

$$\begin{aligned}
(f^{-1})^{-1} &= \{(x, y) \mid (y, x) \in f^{-1}\} \\
&= \{(x, y) \mid (y, x) \in \{(z, w) \mid (w, z) \in f\}\} \\
&= \{(x, y) \mid (x, y) \in f\} \\
&= f.
\end{aligned}$$

That is, $(f^{-1})^{-1} = f$. Since f is a function, we have shown that the inverse of f^{-1} is a function (specifically, f)–that is, we have shown that f^{-1} is invertible. Accordingly, f^{-1} is bijective.

5.14. Let $f : X \to Y$ be a function and suppose that the function $g : Y \to X$ is the inverse of f. Show that the composition $g \circ f : X \to X$ is the identity function, i.e. that $g(f(x)) = x$ for every $x \in X$.

Let $f : X \to Y$ be a function and suppose that the function $g : Y \to X$ is the inverse of f. This implies that f is an invertible function, as we have supposed that the inverse of f is itself a function (and not merely a relation). Since $g = \{(y, x) \mid (x, y) \in f\}$, then $g(f(x)) = z$ implies $(f(x), z) \in g$. But this then implies that $(z, f(x)) \in f$. Since f is necessarily injective (as it is invertible), this implies that $z = x$, as the contrary would yield that $f(x) = f(z)$ with $x \neq z$. Accordingly, $g(f(x)) = x$ for all $x \in X$ and $g \circ f$ is the identity function on X.