HOMEWORK 4

CHAPTER 4: RELATIONS

4.1. Determine whether or not each of the following relations is reflexive, symmetric, antisymmetric, and/or transitive. Are any of these equivalence relations?

(i) The relation $\leq$ on $\mathbb{R}$.

(ii) Equality on $\mathbb{R}$, i.e. the set of ordered pairs of real numbers whose first and second coordinates are equal.

(iii) The relation $\mid$ on $\mathbb{N}$, where $a \mid b$ means that $b = an$ for some $n \in \mathbb{N}$.

(iv) The relation $\sim$ on $\mathbb{N}$ defined by $a \sim b$ if there is an integer $n > 1$ that evenly divides both a and b.

(i) The relation $\leq$ on $\mathbb{R}$ is reflexive, antisymmetric, and transitive, but is not symmetric. Therefore, $\leq$ is *not* an equivalence relation. To see this, observe the following:

  (a) **Reflexive:** Yes. For all $a \in \mathbb{R}$, $a \leq a$.

  (b) **Symmetric:** No. For example, $3 \leq 4$ but $4 \not\leq 3$.

  (c) **Antisymmetric:** Yes. If $a \leq b$ and $b \leq a$, then $a = b$.

  (d) **Transitive:** Yes. If $a \leq b$ and $b \leq c$, then $a \leq c$.

(ii) Equality on $\mathbb{R}$ (i.e. the set of ordered pairs of real numbers whose first and second coordinates are equal) is reflexive, symmetric, antisymmetric, and transitive. Therefore, it is an equivalence relation. To see this, observe the following:

  (a) **Reflexive:** Yes. $a = a$ for all $a \in \mathbb{R}$.

  (b) **Symmetric:** Yes. If $a = b$, then $b = a$.

  (c) **Antisymmetric:** Yes. If $a = b$ and $b = a$, then certainly $a = b$.

  (d) **Transitive:** Yes. If $a = b$ and $b = c$, then $a = c$.

(iii) The relation $|$ on $\mathbb{N}$, where $a \mid b$ means that $b = an$ for some $n \in \mathbb{N}$, is reflexive, antisymmetric, and transitive, but is not symmetric. Therefore, $|$ is *not* an equivalence relation. To see this, observe the following:

(a) **Reflexive:** Yes. $a \mid a$ because $a = (a)(1)$.

(b) **Symmetric:** No. $2 \mid 4$ (since $4 = (2)(2)$, but $4 \nmid 2$ (since $2 \neq 4n$ for any $n \in \mathbb{N}$).

(c) **Antisymmetric:** Yes. If $a \mid b$ and $b \mid a$, then $a = b$. To see this, observe that $a \mid b$ tells us $b = an$ for some $n \in \mathbb{N}$, while $a \mid b$ tells us $a = bm$ for some $m \in \mathbb{N}$. So, $b = an = bmn$, which implies that $mn = 1$ and, correspondingly, that $m = 1$ and $n = 1$. That is, $b = an = (a)(1) = a$.

(d) **Transitive:** Yes. If $a \mid b$ and $b \mid c$, then $a \mid c$. To see this, observe that $a \mid b$ tells us $b = an$ for some $n \in \mathbb{N}$, while $b \mid c$ tells us $c = bm$ for some $m \in \mathbb{N}$. So, $c = bm = (an)m = a(nm)$, which tells us that $a \mid c$.

(iv) The relation $\sim$ on $\mathbb{N}$, defined by $a \sim b$ if there is an integer $n > 1$ that evenly divides both a and b, is symmetric, but is not reflexive, not antisymmetric, and not transitive. Therefore, $\sim$ is *not* an equivalence relation. To see this, observe the following:

(a) **Reflexive:** No. For example, $1 \nsim 1$, since there is no $n > 1$ dividing 1.

(b) **Symmetric:** Yes. If $n > 1$ divides both a and b, then it divides both b and a.

(c) **Antisymmetric:** No. For instance, $2 \sim 4$ (since 2 is an integer greater than one that evenly divides both 2 and 4) and $4 \sim 2$ (again, since 2 is an integer greater than one that evenly divides both 4 and 2), but $2 \neq 4$.

(d) **Transitive:** No. For example, $2 \sim 10$ (since 2 is an integer greater than one that evenly divides both 2 and 10) and $10 \sim 15$ (since 5 is an integer greater than one that evenly divides both 10 and 15), but $2 \nsim 15$ (since the greatest common divisor of 2 and 15 is 1).

4.2. Define the relation $\equiv_5$ on $\mathbb{Z}$ by: $a \equiv_5 b$ if there is an integer n so that $b - a = 5n$. Show that $\equiv_5$ is an equivalence relation on $\mathbb{Z}$. Note: this is a fairly common equivalence relation. The phrase $a \equiv_5 b$ is usually read "a is equivalent to b modulo 5."

We must show that the relation $\equiv_5$ on $\mathbb{Z}$, defined by $a \equiv_5 b$ if and only if $b - a = 5n$ for some $n \in \mathbb{Z}$ is an equivalence relation. That is, we will verify that $\equiv_5$ is reflexive, symmetric, and transitive.

For reflexivity, let $a \in \mathbb{Z}$. Then, we have $a - a = 0 = (5)(0)$. Thus $a \equiv_5 a$ for all $a \in \mathbb{Z}$, and $\equiv_5$ is a reflexive relation.

For symmetry, suppose $a \equiv_5 b$. Then $b - a = 5n$ for some integer n. Multiplying both sides by -1 gives $a - b = 5(-n)$, so $a \equiv_5 b$ implies $b \equiv_5 a$. Hence, $\equiv_5$ is a symmetric relation.

For transitivity, suppose $a \equiv_5 b$ and $b \equiv_5 c$. Then, there exist integers n and m such that $b - a = 5n$ and $c - b = 5m$. Adding these equations gives

$$c - a = (c - b) + (b - a) = 5m + 5n = 5(m + n).$$

As $m+n$ is an integer, we may conclude $a \equiv_5 c$. Therefore, $\equiv_5$ is a transitive relation.

As we have shown that $\equiv_5$ is reflexive, symmetric, and transitive, we may conclude that it is indeed an equivalence relation on $\mathbb{Z}$.

4.3. A relation on a set X is said to be a *partial order* on X if it is reflexive, antisymmetric, and transitive. Let $X = \mathcal{P}(\mathbb{R})$ and consider the inclusion relation defined by $A \subset B$.

(i) Show that $\subset$ is a partial order on X.

(ii) Is strict inclusion $\subsetneq$ a partial order on X?

(i) We show that $\subset$ is a partial order on $\mathcal{P}(\mathbb{R})$. That is, we show $\subset$ is reflexive, antisymmetric, and transitive.
For reflexivity, observe that for every $A \in \mathcal{P}(\mathbb{R})$, we have $A \subset A$ (since $a \in A$ implies $a \in A$). Thus, $\subset$ is a reflexive relation.
For antisymmetry, suppose $A, B \in \mathcal{P}(\mathbb{R})$ with $A \subset B$ and $B \subset A$. Then, each element of A is in B (i.e. $a \in A \Rightarrow a \in B$), and each element of B is in A (i.e. $b \in B \Rightarrow b \in A$), so $A = B$ (in fact, this is a theorem we proved together in class!). Therefore, $\subset$ is an antisymmetric relation.
For transitivity, suppose $A, B, C \in \mathcal{P}(\mathbb{R})$ with $A \subset B$ and $B \subset C$. Then, every element of A is in B and hence also in C, so $A \subset C$. Therefore, $\subset$ is a transitive relation.
As we have shown that $\subset$ is reflexive, antisymmetric, and transitive, we may conclude that it is indeed a partial order on $\mathcal{P}(\mathbb{R})$.

(ii) We show that strict inclusion $\subsetneq$ is not a partial order on $\mathcal{P}(\mathbb{R})$. To do so, recall that $A \subsetneq B$ means $A \subset B$ and $A \neq B$. This is enough to show that $\subsetneq$ is not reflexive (and, therefore, not a partial order). That is, for any subset $A \in \mathcal{P}(\mathbb{R})$, we do *not* have $A \subsetneq A$ since $A \neq A$ is false. Hence, $\subsetneq$ is not reflexive; therefore, it is not a partial order.

Side Note: While $\subsetneq$ is not a partial order, it is indeed anytisymmetric and transitive. For antisymmetry, this property holds vacuously. That is, antisymmetry requires that $A \subsetneq B$ and $B \subsetneq A$ implies $A = B$. But, if $A \subsetneq B$, then $A \neq B$, so it is impossible to have both $A \subsetneq B$ and $B \subsetneq A$ simultaneously. That is, $(A \subsetneq B) \wedge (B \subsetneq A)$ is always false and this is the premise of the implication defining antisymmetry, which means that the implication holds (since $F \Rightarrow P$ is true for any proposition P). For transitivity, we have the property non-vacuously: If $A \subsetneq B$ and $B \subsetneq C$, then $A \subset C$ and $A \neq C$, so $A \subsetneq C$.

4.4. Define a relation $\sim$ on $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ by $(x,y) \sim (w,z)$ if $x^2 + y^2 = w^2 + z^2$. Show that $\sim$ is an equivalence relation on $\mathbb{R}^2$.

We show that the relation $\sim$ on $\mathbb{R}^2$, defined by $(x,y) \sim (w,z)$ if $x^2+y^2 = w^2+z^2$ is an equivalence relation on $\mathbb{R}^2$. To do so, we verify the reflexivity, symmetry, and transitivity of $\sim$.
For reflexivity, let $(x,y) \in \mathbb{R}^2$. Then, certainly we have $x^2+y^2 = x^2+y^2$, so $(x,y) \sim (x,y)$. Therefore, $\sim$ is reflexive.
For symmetry, suppose $(x,y) \sim (w,z)$. Then $x^2+y^2 = w^2+z^2$. As the equality relation on the set of real numbers is certainly an equivalence relation, it is symmetric and $w^2+z^2 = x^2+y^2$.

So, $(w,z) \sim (x,y)$, and $\sim$ is indeed symmetric.
For transitivity, suppose $(x,y) \sim (w,z)$ and $(w,z) \sim (u,v)$. Then. $x^2+y^2 = w^2+z^2$ and $w^2+z^2 = u^2+v^2$. As the equality relation on the set of real numbers is certainly an equivalence relation, it is transitive, so $x^2+y^2 = u^2+v^2$. Therefore, $(x,y) \sim (u,v)$, and $\sim$ is indeed transitive.
As we have shown that $\sim$ is reflexive, symmetric, and transitive, we may conclude that it is indeed an equivalence relation on $\mathbb{R}^2$.

4.5. Consider the equivalence relation defined in Exercise 4.4. For each point $(x,y) \in \mathbb{R}^2$, the equivalence class $[(x,y)]$ is a familiar geometric figure in $\mathbb{R}^2$. What is it?

For each point $(x,y) \in \mathbb{R}^2$, the equivalence class $[(x,y)]$ is a circle centered at the point $(0,0)$ (allowing here for the pathological case of a circle with radius zero to be represented by the single point $(0,0)$, which is the sole member of the equivalence class $[(0,0)]$). That is, writing r to denote the value $\sqrt{x^2+y^2}$, the equivalence class $[(x,y)]$ consists of the set of points $(w,z) \in \mathbb{R}^2$ such that $w^2+z^2 = r^2$. This is simply the set of points that lie on a circle of radius r centered at $(0,0)$.