HOMEWORK 3

CHAPTER 3: SET THEORY

Note: Many of the solutions provided here have much more detail than is necessary. This is done to make it clear what is going on when you check your work–your answers should capture the same general arguments (and in a complete, logically valid way!), but they need not be nearly as verbose.

3.1. Here are some common infinite sets:
$P = \{n \mid n \text{ is prime}\}$ is the set of all prime numbers.
$E = \{n \mid n = 2k \text{ for some } k \in \mathbb{Z}\}$ is the set of all even integers.

How would you write the set of all odd integers in set builder notation? What about the set of all integer powers of 2?

There are many different ways to write the set of all odd integers in set builder notation. Here are a few:

$$\{n \in \mathbb{Z} \mid \exists k \in \mathbb{Z} \;\; n = 2k+1\}$$

$$\{2k+1 \mid k \in \mathbb{Z}\}$$

$$\{n \in \mathbb{Z} \mid \nexists k \in \mathbb{Z} \;\; n = 2k\}$$

$$\{n \in \mathbb{Z} \mid n \notin E\}$$

Likewise, there are many ways to write the set of all integer powers of 2 in set builder notation. Here are two:

$$\left\{n \in \mathbb{Z} \mid \exists k \in \mathbb{Z} \;\; n = 2^k\right\}$$

$$\left\{2^k \mid k \in \mathbb{Z}\right\}$$

3.2. Let $A = \{1, 2, \{1,2\}, \{1,3\}, 4\}$. Determine whether each of the following is en element of A, a subset of A, both an element of A and a subset of A, or neither an element of A nor a subset of A.

(i) 1

(ii) 3

(i) The integer 1 is an element of A. It is not a subset of A since it is not a set (however, the set $\{1\}$ is a subset of A).

(ii) The integer 3 is not an element of A nor a subset of A. However, it is an element of an element of A (namely, 3 is an element of the set $\{1,3\}$, which is an element of A).

3.3. Let A and B be sets and suppose you know that A is not a subset of B. Which of the following is necessarily true? Choose all correct responses.

(i) If $x \in A$, then $x \notin B$.

(ii) If $x \in B$, then $x \in A$.

(iii) There is an element $x \in A$ so that $x \notin B$.

(iv) There is an element $x \in B$ so that $x \notin A$.

Suppose that A is not a subset of B.

(i) The statement "If $x \in A$, then $x \notin B$." is not necessarily true. To see this, consider $A = \{1,2,3\}$ and $B = \{1,2,4\}$. Then, A is not a subset of B (nor is B a subset of A), yet the integers 1 and 2 are elements of both A and B. So, it is indeed possible to have sets A and B for which A is not a subset of B that do not satisfy this statement.

(ii) The statement "If $x \in B$, then $x \in A$." is not necessarily true. To see this, consider $A = \{1,2,3\}$ and $B = \{1,2,4\}$. Then, A is not a subset of B (nor is B a subset of A). Taking $4 \in B$, notice that $4 \notin A$. So, it is indeed possible to have sets A and B for which A is not a subset of B that do not satisfy this statement.

(iii) The statement "There is an element $x \in A$ so that $x \notin B$." is necessarily true. To see this, we can produce a short proof by contradiction to the statement here, which can be stated formally as $\exists x \in A(x \notin B)$. That is, suppose A is not a subset of B and $\neg\big(\exists x \in A(x \notin B)\big)$ is true. Then, we have the following:

$$\neg\big(\exists x \in A(x \notin B)\big) \Leftrightarrow \forall x \in A\big(\neg(x \notin B)\big) \Leftrightarrow \forall x \in A(x \in B).$$

But this says that $A \subseteq B$, contradicting our assumption that A is not a subset of B. Therefore, the original statement must be true, as assuming otherwise produces a contradiction.

(iv) The statement "There is an element $x \in B$ so that $x \notin A$." is not necessarily true. To see this, consider $A = \{1,2,3,4\}$ and $B = \{1,2,4\}$. Then, A is not a subset of B. However, B is a subset of A, so every element of B belongs to A. So, it is indeed possible to have sets A and B for which A is not a subset of B that do not satisfy this statement.

3.4. Let A and B be sets and suppose that $x \notin A \cup B$. Which of the following is necessarily true? Choose all correct responses.

(i) $x \notin A$ or $x \notin B$

(ii) $x \notin A$ and $x \notin B$

(iii) $x \in A$ or $x \in B$

(iv) $x \in A$ and $x \in B$

Let A and B be sets and suppose that $x \notin A \cup B$.

(i) The statement "$x \notin A$ or $x \notin B$" is necessarily true. To see this, first notice that $x \notin A$ or $x \notin B$ is logically equivalent to $\neg(x \in A) \vee \neg(x \in B)$, which is itself equivalent to $\neg\big((x \in A) \wedge (x \in B)\big)$. So, let us prove the statement is true by contradiction. That is, assume the negative of this statement, which is $(x \in A) \wedge (x \in B)$. Then, $x \in A$ and $x \in B$. So, it is certainly the case that $x \in A \cup B$ since, by definition, $A \cup B = \{z \mid z \in A \vee z \in B\}$. But this is a contradiction, as we began by assuming that $x \notin A \cup B$. Therefore, the original statement must be true, as assuming otherwise produces a contradiction.

(ii) The statement "$x \notin A$ and $x \notin B$" is necessarily true. To see this, first notice that $x \notin A$ and $x \notin B$ is logically equivalent to $\neg(x \in A) \wedge \neg(x \in B)$, which is itself equivalent to $\neg\big((x \in A) \vee (x \in B)\big)$. So, let us prove the statement is true by contradiction. That is, assume the negative of this statement, which is $(x \in A) \vee (x \in B)$. Then, $x \in A \cup B$ since, by definition, $A \cup B = \{z \mid z \in A \vee z \in B\}$. But this is a contradiction, as we began by assuming that $x \notin A \cup B$. Therefore, the original statement must be true, as assuming otherwise produces a contradiction.

(iii) The statement "$x \in A$ or $x \in B$" is not necessarily true. In fact, it is necessarily false. Notice that if $(x \in A) \vee (x \in B)$ is true, then $x \in A \cup B$ since, by definition, $A \cup B = \{z \mid z \in A \vee z \in B\}$.

(iv) The statement "$x \in A$ and $x \in B$" is not necessarily true. In fact, it is necessarily false. Notice that if $(x \in A) \wedge (x \in B)$ is true, then $x \in A$ and $x \in B$, so certainly $x \in A \cup B$ since, by definition, $A \cup B = \{z \mid z \in A \vee z \in B\}$.

3.5. Let A and B be sets and suppose that $x \notin A \cap B$. Which of the following is necessarily true? Choose all correct responses.

(i) $x \notin A$ or $x \notin B$

(ii) $x \notin A$ and $x \notin B$

(iii) $x \in A$ or $x \in B$

(iv) $x \in A$ and $x \in B$

Let A and B be sets and suppose that $x \notin A \cap B$.

(i) The statement "$x \notin A$ or $x \notin B$" is necessarily true. To see this, observe that the statement "$x \notin A$ or $x \notin B$" is logically equivalent to $\neg(x \in A) \vee \neg(x \in B)$, which is itself logically equivalent to $\neg\big((x \in A) \wedge (x \in B)\big)$. So, let us prove the statement is true by contradiction. That is, assume the negative of this statement, which is $(x \in A) \wedge (x \in B)$. Then, $x \in A \cap B$ since, by definition, $A \cap B = \{z \mid z \in A \wedge z \in B\}$. But this is a contradiction, as we began by assuming that $x \notin A \cap B$. Therefore, the original statement must be true, as assuming otherwise produces a contradiction.

(ii) The statement "$x \notin A$ and $x \notin B$" is not necessarily true. To see this, consider the example where $A = \{1, 2, 3\}$, $B = \{3, 4, 5\}$, and $x = 1$. Then, since $A \cap B = \{3\}$, $x \notin A \cap B$. But, $x \in A$ and $x \notin B$, so $(x \notin A) \wedge (x \notin B)$ is false.

(iii) The statement "$x \in A$ or $x \in B$" is not necessarily true. To see this, consider the example where $A = \{1, 2, 3\}$, $B = \{3, 4, 5\}$, and $x = 6$. Since $A \cap B = \{3\}$, $x \notin A \cap B$. Indeed, $x \notin A$ and $x \notin B$. But, observe this is logically equivalent to $\neg(x \in A) \wedge \neg(x \in B)$ which is, itself, logically equivalent to $\neg\big((x \in A) \vee (x \in B)\big)$. That is, "$x \in A$ or $x \in B$" is false.

(iv) The statement "$x \in A$ and $x \in B$" is not necessarily true. In fact, it is necessarily false. To see this, observe that, by definition, $A \cap B = \{z \mid z \in A \wedge z \in B\}$. So, $x \notin A \cap B$ requires that $\neg\big((x \in A) \wedge (x \in B)\big)$. That is, $x \notin A \cap B$ requires that "$x \in A$ and $x \in B$" is false.

3.6. Define the following sets: $A = \{1, 3, 9, 27\}$, $B = \{1, 2, 4, 8\}$, $P = \{n \mid n \text{ is a prime integer}\}$, and $E = \{n \mid n = 2k \text{ for some } k \in \mathbb{N}\}$. Find each of the following:

(i) $A \cup B$

(ii) $A \cap B$

(iii) $P \cap E$

(i) $A \cup B = \{1, 2, 3, 4, 8, 9, 27\}$

(ii) $A \cap B = \{1\}$

(iii) $P \cap E = \{2\}$

3.7. For any sets A and B, prove that:

(i) $A \subset A \cup B$

(ii) $A \cap B \subset A$

(iii) $A \cap \emptyset = \emptyset$

(i) We show that $A \subset A \cup B$ for any sets A and B. To do so, let $x \in A$. By definition, $A \cup B = \{z \mid z \in A \vee z \in B\}$. So, $x \in A$ implies (via addition) that $x \in A \vee x \in B$ is certainly true as well. This tells us that $x \in A \cup B$ must also be true. As x was chosen arbitrarily from A, it follows $A \subseteq A \cup B$.

(ii) We show that $A \cap B \subset A$ for any set A. To do so, let $x \in A \cap B$. By definition, $A \cap B = \{z \mid z \in A \wedge z \in B\}$. So, $x \in A \cap B$ requires that $x \in A \wedge x \in B$ is true. By simplification, this tells us that $x \in A$ must also be true. As x was chosen arbitrarily from $A \cap B$, it follows $A \cap B \subset A$.

(iii) We show that $A \cap \emptyset = \emptyset$. To do so, suppose not. That is, suppose that $A \cap \emptyset \neq \emptyset$. Then, there exists some element x such that $x \in A \cap \emptyset$. Then, by definition, $A \cap \emptyset = \{z | z \in A \wedge z \in \emptyset\}$, so this requires that $x \in \emptyset$. But this is a contradiction, as the empty set is the set with no elements. Therefore, the original statement must be true, as assuming otherwise produces a contradiction.

3.8. Let A, B, and C be sets. Prove that $A \subset B \cap C$ if and only if $A \subset B$ and $A \subset C$.

First, suppose $A \subset B \cap C$. We show that this implies $A \subset B$ and $A \subset C$. To do so, let $x \in A$. Then, since $A \subset B \cap C$, certainly $x \in B \cap C$. By definition, $B \cap C = \{z | z \in B \wedge z \in C\}$. So, $x \in B$ and $x \in C$. Since x was chosen arbitrarily from A, we may conclude that $A \subset B$ and $A \subset C$ as desired.

Now suppose that $A \subset B$ and $A \subset C$. We show that this implies $A \subset B \cap C$. To do so, $x \in A$. Then, since $A \subset B$ and $A \subset C$, certainly $x \in B$ and $x \in C$. By definition, $B \cap C = \{z | z \in B \wedge z \in C\}$. So, $x \in B \cap C$ as desired.

3.9. Let A, B, and C be sets. Prove that if $A \subset B \cup C$ and $A \cap B = \emptyset$, then $A \subset C$.

Suppose $A \subset B \cup C$ and $A \cap B = \emptyset$. Let $x \in A$. Then, since $A \subset B \cup C$, certainly $x \in B \cup C$ as well. Moreover, $x \notin B$, as the contrary would imply $x \in A \cap B$ and contradict our assumption that $A \cap B = \emptyset$. By definition, $B \cup C = \{z \mid z \in B \vee z \in C\}$, so $x \in B \cup C$ implies $x \in B \vee x \in C$ must be true. As we have shown that $\neg(x \in B)$ is true as well, we may conclude $x \in C$ by a disjunctive syllogism. Hence, as x was chosen arbitrarily from A, we may conclude $A \subset C$.

3.10. Let A and B be sets in a universal set U. Prove the following:

(i) $A \setminus B = A \cap B'$

(ii) $A \setminus B = A$ if and only if $A \cap B = \emptyset$.

(iii) $A \setminus B = \emptyset$ if and only if $A \subset B$.

(i) Let A and B be sets in a universal set U. Let $x \in A \setminus B$. By definition, $A \setminus B = \{a \mid a \in A \wedge a \notin B\}$, so $x \in A$ and $x \notin B$. Likewise, by definition, $B' = \{u \in U | u \notin B\}$, so we may conclude that $x \in A$ and $x \in B'$. That is, $x \in A \cap B' = \{x \mid x \in A \wedge x \in B'\}$. Therefore, $A \setminus B \subseteq A \cap B'$. Now, suppose $y \in A \cap B'$. Then, $y \in A \wedge y \in B'$ is true. Again, by the definition of B', this implies $y \in A \wedge y \notin B$ is also true, which says $y \in A \setminus B$. Therefore, $A \cap B' \subseteq A \setminus B$. As we have shown containment in both directions, we may conclude $A \cap B' = A \setminus B$.

(ii) Let A and B be sets in a universal set U. We show $A \setminus B = A$ if and only if $A \cap B = \emptyset$. First, for the forward direction, suppose $A \setminus B = A$. That is, suppose $\{x \in U \mid x \in A \wedge x \notin B\} = A$. By way of contradiction, suppose $A \cap B \neq \emptyset$; that is, suppose there exists $x \in A \cap B$. Then, $x \in A \wedge x \in B$ is true. Hence, $x \notin A \setminus B = \{x \in U \mid x \in A \wedge x \notin B\}$. But $A \setminus B = A$, so we have both $x \in A$ and $x \notin A$, which is false by the law of excluded middle. Therefore, $A \cap B = \emptyset$.
Now, for the reverse direction, suppose $A \cap B = \emptyset$. We show this implies $A \setminus B = A$ by showing that $A \setminus B \subseteq A$ and $A \subseteq A \setminus B$. First, let $x \in A \setminus B$. By definition, $A \setminus B = \{y \in U \mid y \in A \wedge y \notin B\}$, so $x \in A$. Hence, $A \setminus B \subseteq A$. Now, suppose instead that $x \in A$. Since $A \cap B = \emptyset$, this implies that $x \notin B$ (as the contrary would yield $x \in A \cap B$). So $x \in A \setminus B = \{y \in U \mid y \in A \wedge y \notin B\}$. Hence, $A \subseteq A \setminus B$. As we have shown the inclusion holds in both directions, we may conclude that $A \setminus B = A$.

(iii) Let A and B be sets in a universal set U. We show that $A \setminus B = \emptyset$ if and only if $A \subset B$. For the forward direction, suppose $A \setminus B = \emptyset$. That is, suppose $\{x \in U \mid x \in A \wedge x \notin B\} = \emptyset$. Suppose $x \in A$. We show that this implies $x \in B$ as well by way of contradiction. That is, suppose $x \notin B$. Then, $x \in A \wedge x \notin B$. So, $x \in A \setminus B$. But this is a contradiction, since $A \setminus B = \emptyset$. Hence, $x \in B$. As x was chosen arbitrarily, we may conclude that $A \subseteq B$.
For the reverse direction, suppose $A \subseteq B$. Then, $A \cap B = A$. Using the fact that $(B')' = B$, part (i) of this problem tells us that $A = A \cap (B')' = A \setminus B'$. By part (ii) of this problem, this then implies that $A \cap B' = \emptyset$. But $A \cap B' = A \setminus B$ by definition. Therefore, we have shown $A \setminus B = \emptyset$ as desired.

3.11. Determine whether or not each of the following is true. If so, provide a proof. If not, provide a counterexample.

(i) $(A \cup B) \times (C \cup D) = (A \times C) \cup (B \times D)$

(ii) $(A \cap B) \times (C \cap D) = (A \times C) \cap (B \times D)$

(i) Let $A = \{1\}$, $B = \{2\}$, $C = \{3\}$, and $D = \{4\}$. Then,

$$\begin{aligned}(A \cup B) \times (C \cup D) &= \big(\{1\} \cup \{2\}\big) \times \big(\{3\} \cup \{4\}\big)\\ &= \{1,2\} \times \{3,4\}\\ &= \{(1,3),(1,4),(2,3),(2,4)\}\end{aligned}$$

while, on the other hand,

$$\begin{aligned}(A \times C) \cup (B \times D) &= (\{1\} \times \{3\}) \cup (\{2\} \times \{4\})\\ &= \{(1,3)\} \cup \{(2,4)\}\\ &= \{(1,3),(2,4)\}\end{aligned}$$

Clearly, these are not equal, so this is indeed a counterexample to the statement provided.

(ii) Suppose $(x, y) \in (A \cap B) \times (C \cap D)$. Then, $x \in A \cap B$ and $y \in C \cap D$ by the definition of the product. Since $A \cap B = \{z \mid z \in A \wedge z \in B\}$ and $C \cap D = \{z \mid z \in C \wedge z \in D\}$, we may also conclude that $x \in A$, $x \in B$, $y \in C$, and $y \in D$. Hence, $(x, y) \in A \times C$ and $(x, y) \in B \times D$. That is, $(x, y) \in (A \times C) \cap (B \times D)$. As x was chosen arbitrarily, we may conclude that $(A \cap B) \times (C \cap D) \subseteq (A \times C) \cap (B \times D)$.
Now, for the reverse inclusion, suppose $(x, y) \in (A \times C) \cap (B \times D)$. Then, $(x, y) \in A \times C$ and $(x, y) \in B \times D$ by the definition of the intersection. By the definition of the product, $(x, y) \in A \times C$ tells us that $x \in A$ and $y \in C$, while $(x, y) \in B \times D$ tells us that $x \in B$ and $y \in D$. Hence, $x \in A \cap B$ and $y \in C \cap D$. That is, $(x, y) \in (A \cap B) \times (C \cap D)$. As (x, y) was chosen arbitrarily, we may conclude that $(A \times C) \cap (B \times D) \subseteq (A \cap B) \times (C \cap D)$.
Therefore, as we have shown the inclusion in both directions, we may conclude $(A \cap B) \times (C \cap D) = (A \times C) \cap (B \times D)$ as desired

3.12. Let A and B be sets.

(i) Show that $\mathcal{P}(A) \cup \mathcal{P}(B) \subset \mathcal{P}(A \cup B)$.

(ii) Show that $\mathcal{P}(A \cup B)$ is not generally a subset of $\mathcal{P}(A) \cup \mathcal{P}(B)$.

(i) Let $S \in \mathcal{P}(A) \cup \mathcal{P}(B)$. Then, either $S \in \mathcal{P}(A)$ or $S \in \mathcal{P}(B)$ by the definition of the union. By the definition of the powerset, this implies that either $S \subseteq A \vee S \subseteq B$. That is, $\forall s \in S \;\; (s \in A \vee s \in B)$. Since $A \cup B = \{x \mid x \in A \vee x \in B\}$, we may restate this as $\forall s \in S \;\; (s \in A \cup B)$. But this then implies $S \subseteq A \cup B$, which means $S \in \mathcal{P}(A \cup B)$ as desired.

(ii) Consider the sets $A = \{a\}$ and $B = \{b\}$. Then, $\mathcal{P}(A) = \{S \mid S \subseteq A\} = \{\emptyset, A\}$ and $\mathcal{P}(B) = \{S \mid S \subseteq B\} = \{\emptyset, B\}$. So, $\mathcal{P}(A) \cup \mathcal{P}(B) = \{\emptyset, A, B\}$. On the other hand, $A \cup B = \{a, b\}$, so $\mathcal{P}(A \cup B) = \{S \mid S \subseteq A\} = \{\emptyset, A, B, A \cup B\}$. Clearly, $A \cup B = \{a, b\} \notin \mathcal{P}(A) \cup \mathcal{P}(B)$, so this is indeed demonstrates that $\mathcal{P}(A \cup B)$ is not generally a subset of $\mathcal{P}(A) \cup \mathcal{P}(B)$.