# HOMEWORK 2

# CHAPTER 2: LOGICAL ARGUMENTS

**2.1.** Show that the following argument is valid:

$$\begin{array}{l} P \\ \neg Q \\ \underline{P \Rightarrow (Q \vee R)} \\ \therefore\ R \end{array}$$

**2.2.** Show that the following argument is valid:

$$\begin{array}{l} P \Rightarrow (R \vee S) \\ R \Rightarrow Q \\ \underline{S \Rightarrow Q} \\ \therefore\ P \Rightarrow Q \end{array}$$

**2.3.** Show that the following argument is valid:

$$\begin{array}{l} \forall x\ (P(x) \Rightarrow Q(x)) \\ \underline{\exists x\ (Q(x) \Rightarrow R(x))} \\ \therefore\ \exists x\ (P(x) \Rightarrow R(x)) \end{array}$$

**2.4.** Which of the following is the best interpretation of the statement: *Every blue meanie is fuzzy*?

(i) There are blue meanies and at least one of them is fuzzy.

(ii) There are blue meanies and at least two of them are fuzzy.

(iii) There are blue meanies and all of them are fuzzy.

(iv) There may not be any blue meanies, but if there are at least one of them is fuzzy.

(v) There may not be any blue meanies, but if there are at least two of them are fuzzy.

(vi) There may not be any blue meanies, but if there are all of them are fuzzy.

**2.5.** Which of the following is the best interpretation of the statement: *Some blue meanies are fuzzy*?

(i) There are blue meanies and at least one of them is fuzzy.

(ii) There are blue meanies and at least two of them are fuzzy.

(iii) There are blue meanies and all of them are fuzzy.

(iv) There may not be any blue meanies, but if there are at least one of them is fuzzy.

(v) There may not be any blue meanies, but if there are at least two of them are fuzzy.

(vi) There may not be any blue meanies, but if there are all of them are fuzzy.

**2.6.** Prove the following:

(i) Some prime number is even.

(ii) There is an integer whose square is not positive.

(iii) There is a city $A$ in North Dakota that is further south than Paris, France.

(iv) There are positive integers $a$ and $b$ such that $ab < a + b$.

(v) There are irrational numbers $a$ and $b$ such that the sum $a + b$ is rational.

**2.7.** Show that the following are false:

(i) Every student in this class has green hair.

(ii) For every real number $x$, $x^2 - 1 \geq 0$.

(iii) Every prime number is less than 1000.

(iv) For all irrational numbers $a$ and $b$, the product $ab$ is irrational.

(v) For all prime numbers $n$ and $m$, the sum $n + m$ is composite.

**2.8.** To show that the statement $\forall x\ P(x)$ is false, it suffices to show that there is an $c$ in the domain of $x$ such that $\neg P(c)$; the value $c$ is called a counterexample. Assume that we believe the statement $\exists x\ P(x)$ is false. Can we prove this by finding a $c$ in the domain of $x$ such that $\neg P(c)$? Justify your answer.

**2.9.** Use induction to prove that $\sum\limits_{i=0}^{n} 2^i = (2^{n+1} - 1)$ for every natural number $n$.

**2.10.** Prove that $\sum\limits_{i=1}^{n} i^2 = \dfrac{n(n+1)(2n+1)}{6}$ for every natural number $n$.

**2.11.** Use induction to prove that $2n \leq 2^n$ for every natural number $n$.

**2.12.** It is sometimes true that statements about the natural numbers are not true for small numbers, but are true for all natural numbers that are large enough. We can modify the Principle of Mathematical Induction so that it is applicable in these situations as follows:

Let $P(n)$ be a statement about the natural number $n$.

(i) If $P(a)$ is true, and

(ii) if $P(k) \implies P(k+1)$ for all $k \geq a$,

then $P(n)$ is true for all natural numbers $n \geq a$.

Use this version of induction to prove that $3^n < n!$ for every integer $n \geq 7$.