HOMEWORK 2

CHAPTER 2: LOGICAL ARGUMENTS

2.1. Show that the following argument is valid:

$$\begin{array}{l} \quad P \\ \quad \neg Q \\ \quad \underline{P \Rightarrow (Q \vee R)} \\ \therefore\ R \end{array}$$

$(1) P$	(Hypothesis)
$(2) P \Rightarrow (Q \vee R)$	(Hypothesis)
$(3) Q \vee R$	(Modus Ponens, (1) and (2))
$(4) \neg Q$	(Hypothesis)
$(5) R$	(Disjunctive Syllogism, (3) and (4))

2.2. Show that the following argument is valid:

$$\begin{array}{l} \quad P \Rightarrow (R \vee S) \\ \quad R \Rightarrow Q \\ \quad \underline{S \Rightarrow Q} \\ \therefore\ P \Rightarrow Q \end{array}$$

$(1) R \Rightarrow Q$	(Hypothesis)
$(2) S \Rightarrow Q$	(Hypothesis)
$(3) \neg R \vee Q$	(Disjunctive form of (1))
$(4) \neg S \vee Q$	(Disjunctive form of (2))
$(5) (\neg R \vee Q) \wedge (\neg S \vee Q)$	(Conjunction, (3) and (4))
$(6) (\neg R \wedge \neg S) \vee Q$	(Distribution, (5))
$(7) \neg(R \vee S) \vee Q$	(De Morgan's (6))
$(8) (R \vee S) \Rightarrow Q$	(Disjunctive form of (7))
$(9) P \Rightarrow (R \vee S)$	(Hypothesis)
$(10) Q$	(Hypothetical Syllogism, (8) and (9))

2.3. Show that the following argument is valid:

$$\begin{array}{l} \forall x\ (P(x) \Rightarrow Q(x)) \\ \underline{\exists x\ (Q(x) \Rightarrow R(x))} \\ \therefore\ \exists x\ (P(x) \Rightarrow R(x)) \end{array}$$

$(1) \exists x\ (Q(x) \Rightarrow R(x))$	(Hypothesis)
$(2) Q(y) \Rightarrow R(y)$ for some y	(Existential Instantiation)
$(3) \forall x\ (P(x) \Rightarrow Q(x))$	(Hypothesis)
$(4) P(y) \Rightarrow Q(y)$	(Universal Instantiation, (2) and (3))
$(5) P(y) \Rightarrow R(y)$	(Hypothetical Syllogism, (2) and (4))
$(6) \exists x\ (P(x) \Rightarrow R(x))$	(Existential Generalization, (5))

2.4. Which of the following is the best interpretation of the statement: *Every blue meanie is fuzzy*?

(i) There are blue meanies and at least one of them is fuzzy.

(ii) There are blue meanies and at least two of them are fuzzy.

(iii) There are blue meanies and all of them are fuzzy.

(iv) There may not be any blue meanies, but if there are at least one of them is fuzzy.

(v) There may not be any blue meanies, but if there are at least two of them are fuzzy.

(vi) There may not be any blue meanies, but if there are all of them are fuzzy.

The best interpretation of the statement ”Every blue meanie is fuzzy” is ”There may not be any blue meanies, but if there are all of them are fuzzy.” To understand why, we consider the other options:

(i) ”There are blue meanies and at least one of them is fuzzy.”
Notice, the statement we began with does not assert that blue meanies exist. Nor does it assert that, should they exist, only one of them is fuzzy.

(ii) ”There are blue meanies and at least two of them are fuzzy.”
Notice, the statement we began with does not assert that blue meanies exist. Nor does it assert that, should they exist, that there are at least two of them. Nor does it assert that, should they exist and there are at least two of them, that only two need be fuzzy.

(iii) There are blue meanies and all of them are fuzzy.”
This statement is much closer but, again, the statement we began with does not assert that blue meanies exist.

(iv) ”There may not be any blue meanies, but if there are at least one of them is fuzzy.”
While the statement we began with does not assert the existence of blue meanies, it does indeed specify that any blue meanies, should they exist, must be fuzzy. Certainly this implies the statement here, but the reverse is not true.

(v) ”There may not be any blue meanies, but if there are at least two of them are fuzzy.”
While the statement we began with does not assert the existence of blue meanies, it does indeed specify that any blue meanies, should they exist, must be fuzzy. Certainly this implies the statement here if , but the reverse is not true. Notice, the statement we began with also does not assert that, should blue meanies exist, that there are at least two of them.

2.5. Which of the following is the best interpretation of the statement: *Some blue meanies are fuzzy*?

(i) There are blue meanies and at least one of them is fuzzy.

(ii) There are blue meanies and at least two of them are fuzzy.

(iii) There are blue meanies and all of them are fuzzy.

(iv) There may not be any blue meanies, but if there are at least one of them is fuzzy.

(v) There may not be any blue meanies, but if there are at least two of them are fuzzy.

(vi) There may not be any blue meanies, but if there are all of them are fuzzy.

The best interpretation of the statement ”Some blue meanies are fuzzy” is ”There may not be any blue meanies, but if there are at least one of them is fuzzy.” To understand why, we consider the other options:

(i) ”There are blue meanies and at least one of them is fuzzy.”
This is close, but notice, the statement we began with does not assert that blue meanies exist.

(ii) ”There are blue meanies and at least two of them are fuzzy.”
Notice, the statement we began with does not assert that blue meanies exist. Nor does it assert that, should they exist, that there are at least two of them. Nor does it assert that, should they exist and there are at least two of them, that two need be fuzzy.

(iii) There are blue meanies and all of them are fuzzy."
Notice, the statement we began with does not assert that blue meanies exist. Moreover, it also does not make any claim about all of the blue meanies, should they exist.

(iv) "There may not be any blue meanies, but if there are at least two of them are fuzzy."
While the statement we began with does not assert the existence of blue meanies, it does indeed specify that at least one blue meanie, should such things they exist, must be fuzzy. However, notice that the statement we began with does not assert that, should blue meanies exist, that there are at least two of them, nor does it assert that more than one need be fuzzy.

2.6. Prove the following:

(i) Some prime number is even.

(ii) There is an integer whose square is not positive.

(iii) There is a city A in North Dakota that is further south than Paris, France.

(iv) There are positive integers a and b such that $ab < a + b$.

(v) There are irrational numbers a and b such that the sum $a + b$ is rational.

(i) Some prime number is even.

Proof. We prove the statement directly: Observe the integer 2 has only itself and 1 as factors. Hence, it is prime. By definition, an integer n is even if $n = 2k$ for some integer k. Since $2 = 2(1)$, it is certainly the case that 2 is even. Therefore, we have shown directly that an even prime number exists. $\square$

(ii) There is an integer whose square is not positive.

Proof. We prove the statement directly: Observe the integer 0 is equal to itself when squared–i.e.; $0^2 = 0$. By definition, an integer n is positive if $n > 0$. Since $0^2 = 0 \leq 0$, it is certainly the case that 0^2 is not positive. Therefore, we have shown directly that there is an integer whose square is not positive. $\square$

(iii) There is a city A in North Dakota that is further south than Paris, France.

Proof. We prove the statement directly by providing an example of such a city. First, note that Paris, France is located at approximately 48°51′24″N, 2°21′8″E. There are many towns in North Dakota with a latitude lower than this–for instance, Hankinson, North Dakota is located at approximately 46°04′24″N, 96°53′30″W. That is, Hankinson, North Dakota is further south than Paris, France. Therefore, the statement that there is a city A in North Dakota that is further south than Paris, France is true. $\square$

(iv) There are positive integers a and b such that $ab < a + b$.

Proof. We prove the statement directly: Observe the integers $a = 1$ and $b = 1$ satisfy $1 = 1 * 1 < 1 + 1 = 2$. Therefore, we have shown that there are positive integers a and b such that $ab < a + b$. $\square$

(v) There are irrational numbers a and b such that the sum $a + b$ is rational.

Proof. We prove the statement directly: As we have shown in class, $\sqrt{2}$ is irrational. Now, $-\sqrt{2}$ must also be irrational, as the contrary would imply there exist integers a and b such that $-\sqrt{2} = \frac{a}{b}$, implying that $\sqrt{2} = \frac{-a}{b}$ and contradicting its irrationality. So, both $\sqrt{2}$ and $-\sqrt{2}$ are irrational, but $\sqrt{2} + (-\sqrt{2}) = 0 = \frac{0}{1}$, which is rational. Therefore, we have shown that there are irrational numbers a and b such that the sum $a + b$ is rational. $\square$

2.7. Show that the following are false:

(i) Every student in this class has green hair.

(ii) For every real number x, $x^2 - 1 \geq 0$.

(iii) Every prime number is less than 1000.

(iv) For all irrational numbers a and b, the product ab is irrational.

(v) For all prime numbers n and m, the sum $n + m$ is composite.

(i) Every student in this class has green hair.

Disproof. We provide a counterexample: [insert your name here] does not have green hair and is enrolled in this class. Therefore, the statement is false.

(ii) For every real number x, $x^2 - 1 \geq 0$.

Disproof. We provide a counterexample: Let $x = 0$. Then, $x^2 - 1 = 0^2 - 1 = -1 < 0$. Therefore, the statement is false.

(iii) Every prime number is less than 1000.

Disproof. We provide a counterexample: The number 1009 is prime, but $1009 > 1000$. Therefore, the statement is false.

(iv) For all irrational numbers a and b, the product ab is irrational.

Disproof. We provide a counterexample: Let $a = \sqrt{2}$ and let $b = \sqrt{2}$. As we have shown in lecture, both a and b are irrational. However, $ab = (\sqrt{2})^2 = 2 = \frac{2}{1}$, which is rational. Therefore, the statement is false.

(v) For all prime numbers n and m, the sum $n + m$ is composite.

Disproof. We provide a counterexample: Let $n = 2$ and $m = 3$. Both n and m are prime, but $n + m = 2 + 3 = 5$ is also prime. Therefore, the statement is false.

2.8. To show that the statement $\forall x\ P(x)$ is false, it suffices to show that there is an c in the domain of x such that $\neg P(c)$; the value c is called a counterexample. Assume that we believe the statement $\exists x\ P(x)$ is false. Can we prove this by finding a c in the domain of x such that $\neg P(c)$? Justify your answer.

No, it is not possible to prove $\neg(\exists x\ P(x))$ by showing $\neg P(c)$ (i.e. that $P(c)$ is false) for some value of c in the domain of P. For example, $P(x)$ could be the statement "x is negative" with domain the integers. Then $P(2)$ is false, but $\exists x\ P(x)$ is a true statement (as, for instance, $P(-1)$ demonstrates). That is, a single example of a predicate being false is not enough to show that an existentially quantified statement about that predicate is false–the existential quantifier merely asserts the existence of *some* value for which the predicate is true and does not claim that it is true *at any other* value.

Note: There is a situation where it is possible to show $\neg(\exists x\ P(x))$ by showing $\neg P(c)$, which is the very specific case where the domain of $P(x)$ is the single-element set $\{c\}$. Outside of this, however, the above argument holds.

2.9. Use induction to prove that $\sum_{i=0}^{n} 2^i = (2^{n+1} - 1)$ for every natural number n.

Proof. We prove that $\sum_{i=0}^{n} 2^i = (2^{n+1} - 1)$ for every natural number n via induction.

Base Case: Let $n = 1$. Then,

$$\begin{aligned}\sum_{i=0}^{1} 2^i &= 2^0 + 2^1 \\ &= 1 + 2 \\ &= 3 \\ &= 4 - 1 \qquad\qquad = 2^{1+1} - 1.\end{aligned}$$

Therefore, the statement holds for $n = 1$.

Inductive Step: Suppose the statement holds for some value $k \geq 1$; i.e., that $\sum_{i=0}^{k} 2^i = (2^{k+1} - 1)$. We show that this implies the statement holds for $k+1$ as well; i.e., that $\sum_{i=0}^{k+1} 2^i = (2^{(k+1)+1} - 1)$. To do so, observe the following:

$$\begin{aligned}\sum_{i=0}^{k+1} 2^i &= \left(\sum_{i=0}^{k} 2^i\right) + 2^{k+1} \\ &= (2^{k+1} - 1) + 2^{k+1} \\ &= 2^{k+1} + 2^{k+1} - 1+ \\ &= 2(2^{k+1}) - 1+ \\ &= 2^{(k+1)+1} - 1.\end{aligned}$$

Conclusion: As we have shown the base case of $n = 1$ and that having the statement for k implies the statement for $k+1$, it follows by the principle of mathematical induction that the statement is true for all natural numbers n. $\square$

2.10. Prove that $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$ for every natural number n.

Proof. We prove that $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$ for every natural number n via induction.

Base Case: Let $n = 1$. Then,

$$\begin{aligned}
\sum_{i=1}^{1} i^2 &= 1^1 \\
&= 1 \\
&= \frac{6}{6} \\
&= \frac{(1)(2)(3)}{6} \\
&= \frac{(1)(1+1)(2(1)+1)}{6}.
\end{aligned}$$

Therefore, the statement holds for $n = 1$.

Inductive Step: Suppose the statement holds for some value $k \geq 1$; i.e., that $\sum_{i=1}^{k} i^2 = \frac{k(k+1)(2k+1)}{6}$. We show that this implies the statement holds for $k+1$ as well; i.e., that $\sum_{i=1}^{k+1} i^2 = \frac{(k+1)((k+1)+1)(2(k+1)+1)}{6}$. To do so, observe the following:

$$\begin{aligned}
\sum_{i=1}^{k+1} i^2 &= \left(\sum_{i=1}^{k} i^2\right) + (k+1)^2 \\
&= \left(\frac{k(k+1)(2k+1)}{6}\right) + (k+1)^2 \\
&= \frac{k(k+1)(2k+1)}{6} + \frac{6(k+1)^2}{6} \\
&= \frac{k(k+1)(2k+1) + 6(k+1)^2}{6} \\
&= \frac{(k+1)\big(k(2k+1) + 6(k+1)\big)}{6} \\
&= \frac{(k+1)\big(2k^2 + k + 6k + 6\big)}{6} \\
&= \frac{(k+1)(k+2)(2k+3)}{6} \\
&= \frac{(k+1)((k+1)+1)(2(k+1)+1)}{6}.
\end{aligned}$$

Conclusion: As we have shown the base case of $n = 1$ and that having the statement for k implies the statement for $k+1$, it follows by the principle of mathematical induction that the statement is true for all natural numbers n. □

2.11. Use induction to prove that $2n \leq 2^n$ for every natural number n.

Proof. We prove that $2n \leq 2^n$ for every natural number n via induction.

Base Case: Let $n = 1$. Then, $2(1) = 2 \leq 2 = 2^1$. Therefore, the statement holds for $n = 1$.

Inductive Step: Suppose the statement holds for some value $k \geq 1$; i.e., that $2k \leq 2^k$. We show that this implies the statement holds for $k+1$ as well; i.e., that $2(k+1) \leq 2^{k+1}$. To do so, observe the following:

$$\begin{aligned} 2(k+1) &= 2k+2 \\ &\leq 2^k + 2 \\ &= 2^k + 2^1 \\ &\leq 2^k + 2^k \\ &= 2(2^k) \\ &= 2^{k+1}. \end{aligned}$$

Conclusion: As we have shown the base case of $n = 1$ and that having the statement for k implies the statement for $k+1$, it follows by the principle of mathematical induction that the statement is true for all natural numbers n. □

2.12. It is sometimes true that statements about the natural numbers are not true for small numbers, but are true for all natural numbers that are large enough. We can modify the Principle of Mathematical Induction so that it is applicable in these situations as follows:

Let $P(n)$ be a statement about the natural number n.

(i) If $P(a)$ is true, and

(ii) if $P(k) \implies P(k+1)$ for all $k \geq a$,

then $P(n)$ is true for all natural numbers $n \geq a$.

Use this version of induction to prove that $3^n < n!$ for every integer $n \geq 7$.

Proof. We prove that $3^n < n!$ for every natural number $n \geq 7$ via induction.

Base Case: Let $n = 7$. Then, $3^7 = 2187 < 5040 = 7!$. Therefore, the statement holds for $n = 7$.

Inductive Step: Suppose the statement holds for some value $k \geq 7$; i.e., that $3^k < k!$. We show that this implies the statement holds for $k+1$ as well; i.e., that $3^{k+1} < (k+1)!$. To do so, observe the following:

$$\begin{aligned} 3^{k+1} &= 3(3^k) \\ &< 3(k!) \\ &< (k+1)(k!) \\ &= (k+1)!. \end{aligned}$$

Conclusion: As we have shown the base case of $n = 7$ and that having the statement for k implies the statement for $k+1$, it follows by the principle of mathematical induction that the statement is true for every integer $n \geq 7$. $\square$