HOMEWORK 1

CHAPTER 1: ELEMENTARY LOGIC

1.1. Show that $\neg(\neg P) \Leftrightarrow P$ is a tautology.

To show that $\neg(\neg P) \Leftrightarrow P$ is a tautology, we construct a truth table. First, recall that $A \Leftrightarrow B$ is false if and only if A and B do not share the same truth value. So, substituting $\neg(\neg(P))$ for A and P for B, we get the following truth table (notice, we only need two rows, since there is only one proposition involved):

P	$\neg P$	$\neg(\neg P)$	$\neg(\neg P) \Leftrightarrow P$
T	F	T	T
F	T	F	T

As the right-most column consists only of true values, we conclude that $\neg(\neg P) \Leftrightarrow P$ is a tautology.

1.2. Determine whether each of the following is a tautology, a contradiction, or neither one.

(i) $(P \Rightarrow Q) \Rightarrow Q$

To begin, recall that $A \Rightarrow B$ is false if and only if B is false and A is true. We use this logic to construct a truth table for $(P \Rightarrow Q) \Rightarrow Q$:

P	Q	$P \Rightarrow Q$	$(P \Rightarrow Q) \Rightarrow Q$
T	T	T	T
T	F	F	T
F	T	T	T
F	F	T	F

Hence, $(P \Rightarrow Q) \Rightarrow Q$ is neither a tautology nor a contradiction.

(ii) $(P \wedge (P \Rightarrow Q)) \Rightarrow Q$

To begin, recall that $A \Rightarrow B$ is false if and only if B is false and A is true, while $A \Leftrightarrow B$ is false if and only if A and B do not have the same truth value. We use this logic to construct a truth table for $(P \wedge (P \Rightarrow Q)) \Rightarrow Q$:

P	Q	$P \Rightarrow Q$	$(P \wedge (P \Rightarrow Q))$	$(P \wedge (P \Rightarrow Q)) \Rightarrow Q$
T	T	T	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

As the right-most column consists only of true values, we conclude that $(P \wedge (P \Rightarrow Q)) \Rightarrow Q$ is a tautology.

(iii) $(P \Rightarrow Q) \Leftrightarrow (Q \Rightarrow P)$

To begin, recall that $A \Rightarrow B$ is false if and only if B is false and A is true, while $A \Leftrightarrow B$ is false if and only if A and B do not have the same truth value. We use this logic to construct a truth table for $(P \Rightarrow Q) \Leftrightarrow (Q \Rightarrow P)$:

P	Q	$P \Rightarrow Q$	$Q \Rightarrow P$	$(P \Rightarrow Q) \Leftrightarrow (Q \Rightarrow P)$
T	T	T	T	T
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

Hence, $(P \Rightarrow Q) \Leftrightarrow (Q \Rightarrow P)$ is neither a tautology nor a contradiction.

(iv) $((P \Rightarrow Q) \wedge (Q \Rightarrow R)) \Rightarrow (P \Rightarrow R)$

To begin, recall that $A \Rightarrow B$ is false if and only if B is false and A is true, while $A \wedge B$ is true if and only if A and B are true. We use this logic to construct a truth table for $(P \Rightarrow Q) \Leftrightarrow (Q \Rightarrow P)$:

P	Q	R	$P \Rightarrow Q$	$Q \Rightarrow R$	$(P \Rightarrow Q) \wedge (Q \Rightarrow R)$	$P \Rightarrow R$	$((P \Rightarrow Q) \wedge (Q \Rightarrow R)) \Rightarrow (P \Rightarrow R)$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	T
T	F	T	F	T	F	T	T
T	F	F	F	T	F	F	T
F	T	T	T	T	T	T	T
F	T	F	T	F	F	T	T
F	F	T	T	T	T	T	T
F	F	F	T	T	T	T	T

Hence, $((P \Rightarrow Q) \wedge (Q \Rightarrow R)) \Rightarrow (P \Rightarrow R)$ is a tautology.

(v) $(P \Rightarrow Q) \wedge (P \Rightarrow \neg Q)$

P	Q	$P \Rightarrow Q$	$P \Rightarrow \neg Q$	$(P \Rightarrow Q) \wedge (P \Rightarrow \neg Q)$
T	T	T	F	F
T	F	F	T	F
F	T	T	T	T
F	F	T	T	T

Hence, $(P \Rightarrow Q) \wedge (P \Rightarrow \neg Q)$ is neither a tautology nor a contradiction.

(vi) $P \wedge (P \Rightarrow Q) \wedge (P \Rightarrow \neg Q)$

P	Q	$P \Rightarrow Q$	$P \wedge (P \Rightarrow Q)$	$P \Rightarrow \neg Q$	$(P \wedge (P \Rightarrow Q)) \wedge (P \Rightarrow \neg Q)$
T	T	T	T	F	F
T	F	F	F	T	F
F	T	T	F	T	F
F	F	T	F	T	F

Hence, $P \wedge (P \Rightarrow Q) \wedge (P \Rightarrow \neg Q)$ is a contradiction.

(vii) $(\neg Q \wedge (P \Rightarrow Q)) \Rightarrow \neg P$

P	Q	$P \Rightarrow Q$	$\neg Q$	$(\neg Q \wedge (P \Rightarrow Q))$	$\neg P$	$(\neg Q \wedge (P \Rightarrow Q)) \Rightarrow \neg P$
T	T	T	F	F	F	T
T	F	F	T	F	F	T
F	T	T	F	F	T	T
F	F	T	T	T	T	T

Hence, $(\neg Q \wedge (P \Rightarrow Q)) \Rightarrow \neg P$ is a tautology.

(viii) $((P \vee Q) \wedge \neg Q) \Rightarrow P$

P	Q	$(P \vee Q)$	$\neg Q$	$(P \vee Q) \wedge \neg Q$	$((P \vee Q) \wedge \neg Q) \Rightarrow P$
T	T	T	F	F	T
T	F	T	T	T	T
F	T	T	F	F	T
F	F	F	T	F	T

Hence, $((P \vee Q) \wedge \neg Q) \Rightarrow P$ is a tautology.

(ix) $((P \vee Q) \Rightarrow R) \Rightarrow (P \Rightarrow R)$

P	Q	R	$(P \vee Q)$	$(P \vee Q) \Rightarrow R$	$(P \Rightarrow R)$	$((P \vee Q) \Rightarrow R) \Rightarrow (P \Rightarrow R)$
T	T	T	T	T	T	T
T	T	F	T	F	F	T
T	F	T	T	T	T	T
T	F	F	T	F	F	T
F	T	T	T	T	T	T
F	T	F	T	F	T	T
F	F	T	F	T	T	T
F	F	F	F	T	T	T

Hence, $((P \vee Q) \Rightarrow R) \Rightarrow (P \Rightarrow R)$ is a tautology.

1.3. Determine whether or not each of the following is equivalent to the implication $P \Rightarrow Q$.

(i) The converse: $Q \Rightarrow P$

P	Q	$Q \Rightarrow P$	$P \Rightarrow Q$
T	T	T	T
T	F	T	F
F	T	F	T
F	F	T	T

As the final two columns are not the same, $Q \Rightarrow P$ is not equivalent to $P \Rightarrow Q$.

(ii) The contrapositive: $\neg Q \Rightarrow \neg P$

P	Q	$\neg Q$	$\neg P$	$\neg Q \Rightarrow \neg P$	$P \Rightarrow Q$
T	T	F	F	T	T
T	F	T	F	F	F
F	T	F	T	T	T
F	F	T	T	T	T

As the final two columns are the same, $\neg Q \Rightarrow \neg P$ is equivalent to $P \Rightarrow Q$.

(iii) The inverse: $\neg P \Rightarrow \neg Q$

P	Q	$\neg P$	$\neg Q$	$\neg P \Rightarrow \neg Q$	$P \Rightarrow Q$
T	T	F	F	T	T
T	F	F	T	T	F
F	T	T	F	F	T
F	F	T	T	T	T

As the final two columns are not the same, $\neg P \Rightarrow \neg Q$ is not equivalent to $P \Rightarrow Q$.

1.4. Use truth tables to prove the following:

(i) Disjunction is commutative, i.e. $P \vee Q$ is equivalent to $Q \vee P$.

P	Q	$P \vee Q$	$Q \vee P$
T	T	T	T
T	F	T	T
F	T	T	T
F	F	F	F

As the final two columns are the same, $P \vee Q$ is equivalent to $Q \vee P$.

(ii) Conjunction is commutative.

P	Q	$P \wedge Q$	$Q \wedge P$
T	T	T	T
T	F	F	F
F	T	F	F
F	F	F	F

As the final two columns are the same, $P \wedge Q$ is equivalent to $Q \wedge P$.

(iii) Disjunction is associative, i.e. $P \vee (Q \vee R)$ is equivalent to $(P \vee Q) \vee R$.

P	Q	$Q \vee R$	$P \vee Q$	$P \vee (Q \vee R)$	$(P \vee Q) \vee R$
T	T	T	T	T	T
T	F	F	T	T	T
F	T	T	T	T	T
F	F	F	F	F	F

As the final two columns are the same, $P \vee (Q \vee R)$ is equivalent to $(P \vee Q) \vee R$.

(iv) Conjunction is associative.

P	Q	$Q \wedge R$	$P \wedge Q$	$P \wedge (Q \wedge R)$	$(P \wedge Q) \wedge R$
T	T	F	T	F	F
T	F	F	F	F	F
F	T	F	F	F	F
F	F	F	F	F	F

As the final two columns are the same, $P \vee (Q \vee R)$ is equivalent to $(P \vee Q) \vee R$.

1.5. Show that $P \wedge (Q \vee R)$ is not equivalent to $(P \wedge Q) \vee R$.

P	Q	$Q \vee R$	$P \wedge Q$	$P \wedge (Q \vee R)$	$(P \wedge Q) \vee R$
T	T	T	T	T	T
T	F	F	F	F	F
F	T	T	F	F	F
F	F	F	F	F	F

As the final two columns are not the same, $P \wedge (Q \vee R)$ is not equivalent to $(P \wedge Q) \vee R$.

1.6. Use truth tables for each of the following:

(i) Show that conjunction distributes over disjunction, i.e. that $P \wedge (Q \vee R)$ is equivalent to $(P \wedge Q) \vee (P \wedge R)$.

P	Q	R	$Q \vee R$	$P \wedge Q$	$P \wedge R$	$P \wedge (Q \vee R)$	$(P \wedge Q) \vee (P \wedge R)$
T	T	T	T	T	T	T	T
T	T	F	T	T	F	T	T
T	F	T	T	F	T	T	T
T	F	F	F	F	F	F	F
F	T	T	T	F	F	F	F
F	T	F	T	F	F	F	F
F	F	T	T	F	F	F	F
F	F	F	F	F	F	F	F

As the final two columns are the same, $P \wedge (Q \vee R)$ is equivalent to $(P \wedge Q) \vee (P \wedge R)$.

(ii) Show that disjunction distributes over conjunction.

P	Q	R	$Q \wedge R$	$P \vee Q$	$P \vee R$	$P \vee (Q \wedge R)$	$(P \vee Q) \wedge (P \vee R)$
T	T	T	T	T	T	T	T
T	T	F	F	T	T	T	T
T	F	T	F	T	T	T	T
T	F	F	F	T	T	T	T
F	T	T	T	T	T	T	T
F	T	F	F	T	F	F	F
F	F	T	F	F	T	F	F
F	F	F	F	F	F	F	F

As the final two columns are the same, $P \vee (Q \wedge R)$ is equivalent to $(P \vee Q) \wedge (P \vee R)$.

1.7. Determine whether or not each of the following syllogisms is valid.

(i) Some dogs are beagles. Snoopy is a dog. Hence Snoopy is a beagle. This syllogism is not valid. The first statement specifies that *some* dogs are beagles, not that *all* dogs are beagles. Hence, the information provided by the second statement (i.e. that Snoopy is a dog) does not necessarily imply that Snoopy is a beagle (the third statement).

(ii) All cats are independent. Some pets are cats. Hence some pets are independent. This syllogism is valid. The first statement specifies that *all* cats are independent. Hence, the information provided by the second statement (i.e. that some pets are cats) implies that there exist pets that are independent (the third statement).

(iii) No pigs fly. Porky is a pig. Hence Porky does not fly. This syllogism is valid. The first statement specifies that *all* pigs do not fly. Hence, the information provided by the second statement (i.e. that Porky is a pig) implies that there Porky does not fly (the third statement).

(iv) No orcs are ogres. All ogres are green. Hence no orcs are green. This syllogism is not valid. The first statement specifies that *all* orcs are not ogres. Hence, the information provided by the second statement (i.e. that all ogres are green) does not necessarily imply that all orcs are not green (the third statement)–it is possible that there could be many non-ogre things that are green.

1.8. Negate each of the following:

(i) Every cloud has a silver lining. Let $P(x)$ be the statement "x has a silver lining" so that the sentence in question reads, symbolically, as $\forall x P(x)$, where the domain of discourse the quantifier ranges over is the set of all clouds. Negativing, this becomes $\neg(\forall x P(x)) = \exists x(\neg P(x))$. That is, the negation of this statement is: "Some cloud does not have a silver lining."

(ii) If it's Tuesday, this must be Belgium. Let P be the proposition "It is Tuesday" and let Q be the proposition "This is Belgium" so that the sentence in question reads, symbolically, as $P \Rightarrow Q$. Negativing, this becomes $\neg(P \Rightarrow Q) = P \wedge (\neg Q)$. That is, the negation of this statement is: "It's Tuesday and this isn't Belgium."

(iii) There is a light at the end of every tunnel. Let $P(x)$ be the statement "There is a light at the end of x" so that the sentence in question reads, symbolically, as $\forall x P(x)$, where the domain of discourse the quantifier ranges over is the set of all tunnels. Negativing, this becomes $\neg(\forall x P(x)) = \exists x(\neg P(x))$. That is, the negation of this statement is: "There is not a light at the end of some tunnel."

1.9. Use truth tables to show that:

(i) $\neg(P \wedge Q) \Leftrightarrow (\neg P \vee \neg Q)$

P	Q	$P \wedge Q$	$\neg(P \wedge Q)$	$\neg Q$	$\neg P$	$\neg P \vee \neg Q$	$\neg(P \wedge Q) \Leftrightarrow (\neg P \vee \neg Q)$
T	T	T	F	F	F	F	T
T	F	F	T	T	F	T	T
F	T	F	T	F	T	T	T
F	F	F	T	T	T	T	T

As the final column consists only of true values, $\neg(P \wedge Q) \Leftrightarrow (\neg P \vee \neg Q)$ holds as a valid rule of inference (i.e. it is a tautology).

(ii) $\neg(P \vee Q) \Leftrightarrow (\neg P \wedge \neg Q))$

P	Q	$P \vee Q$	$\neg(P \vee Q)$	$\neg Q$	$\neg P$	$\neg P \wedge \neg Q$	$\neg(P \vee Q) \Leftrightarrow (\neg P \wedge \neg Q))$
T	T	T	F	F	F	F	T
T	F	T	F	T	F	F	T
F	T	T	F	F	T	F	T
F	F	F	T	T	T	T	T

As the final column consists only of true values, $\neg(P \vee Q) \Leftrightarrow (\neg P \wedge \neg Q))$ holds as a valid rule of inference (i.e. it is a tautology).

1.10. Use a truth table to show that $\neg(P \Rightarrow Q) \Leftrightarrow (P \wedge \neg Q)$

P	Q	$P \Rightarrow Q$	$\neg(P \Rightarrow Q)$	$\neg Q$	$P \wedge \neg Q$	$\neg(P \Rightarrow Q) \Leftrightarrow (P \wedge \neg Q)$
T	T	T	F	F	F	T
T	F	F	T	T	T	T
F	T	T	F	F	F	T
F	F	T	F	T	F	T

As the final column consists only of true values, $\neg(P \vee Q) \Leftrightarrow (\neg P \wedge \neg Q))$ holds as a valid rule of inference (i.e. it is a tautology).